New Directions in Non-Relativistic and Relativistic Rotational and Multipole Kinematics for N-Body and Continuous Systems.

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Abstract

In non-relativistic mechanics the center of mass of an isolated system is easily separated out from the relative variables. For a N-body system these latter are usually described by a set of Jacobi normal coordinates, based on the clustering of the centers of mass of sub-clusters. The Jacobi variables are then the starting point for separating *orientational* variables, connected with the angular momentum constants of motion, from *shape* (or *vibrational*) variables. Jacobi variables, however, cannot be extended to special relativity. We show by group-theoretical methods that two new sets of relative variables can be defined in terms of a *clustering of the angular momenta of sub-clusters* and directly related to the so-called *dynamical body frames* and *canonical spin bases.* The underlying group-theoretical structure allows a direct extension of such notions from a non-relativistic to a special-relativistic context if one exploits the *rest-frame instant form of dynamics.* The various known definitions of relativistic center of mass are recovered. The separation of suitable relative variables from the so-called *canonical internal* center of mass leads to the correct kinematical framework for the relativistic theory of the orbits for a N-body system with action-at-a-distance interactions. The rest-frame instant form is also shown to be the correct kinematical framework for introducing the Dixon multi-poles for closed and open N-body systems, as well as for continuous systems, exemplified here by the configurations of the Klein-Gordon field that are compatible with the previous notions of center of mass.

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I. INTRODUCTION.

In nuclear, atomic and molecular physics, as well as in celestial mechanics, an old basic problem is to exploit the translational and rotational invariance of a N-body system for eliminating as many global variables as possible to get a reduced system described by a well defined set of *relative degrees of freedom*. In molecular dynamics, for instance, this reduction is instrumental for the definition of molecular vibrations and rotations. A similar old problem exists for isolated continuous deformable bodies: in this case the usual Euler kinematics of rigid bodies in the body frame must be generalized to understand phenomena like the *falling cat* and the *diver*. We suggest to see the review part of Ref.[1] for an essential selection of the huge bibliography on such issues.

Since most of the applications concerning these problems are typically non-relativistic, nearly all treatments deal with non-relativistic systems of the *kinetic-plus-potential type*. However, developments in particle physics, astrophysics and general relativity suggest to extend the treatment from the absolute Galilei space plus time to Minkowski space-time and then to Einstein space-times. Furthermore, the case of the general relativistic N-body problem, in particular, forces us to simulate the issue of the divergences in the self-energies with a multipolar expansion. We need, therefore, a translation of every result in the language of such kind of expansions.

In this article we show a new method for the treatment of relative variables for a many-body system *both non-relativistic and relativistic*. Our proposal is based upon a conjunction of Hamiltonian and group-theoretical methods leading to a systematic generalization, valid for generic deformable systems, of the standard concept of *body frame* for rigid systems. We also show that our procedure constitutes the proper kinematical framework for introducing multipolar expansions for closed and open systems.

In Newton mechanics isolated systems of N particles possess 3N degrees of freedom in configuration space and 6N in phase space. The Abelian nature of the overall translational invariance, with its associated three commuting Noether constants of the motion, makes possible the decoupling and, therefore, the elimination of either three configurational variables or three pairs of canonical variables, respectively (*separation of the center-of-mass motion*). In this way one is left with either 3N-3 relative coordinates $\vec{\rho}_a$ or 6N-6 relative phase space variables $\vec{\rho}_a$, $\vec{p}_a$, $a = 1,..,N-1$ while the center-of-mass angular momentum or spin is $\vec{S} = \sum_{a=1}^{N-1} \vec{\rho}_a \times \vec{p}_a$. Most of the calculations of the non-relativistic theory employ the sets of $3N-3$ *Jacobi normal relative coordinates* $\vec{s}_a$ (see Ref.[1]) that diagonalize the quadratic form associated with the relative kinetic energy (the spin becomes $\vec{S} = \sum_{a=1}^{N-1} \vec{s}_a \times \vec{\pi}_{sa}$, with $\vec{\pi}_{sa}$ momenta conjugated to the $\vec{s}_a$'s). Each set of Jacobi normal coordinates $\vec{s}_a$ orders the N particles into a *hierarchy of clusters*, in which each cluster of two or more particles has a mass given by an eigenvalue (*reduced mass*) of the quadratic form; the Jacobi normal coordinates join the centers of mass of cluster pairs.

On the other hand, the non-Abelian nature of the overall rotational invariance entails that an analogous intrinsic separation of *rotational* (or *orientational*) configurational variables from others which could be called *shape* or *vibrational* be impossible. As a matter of fact, this is one of the main concerns of molecular physics and of advanced mechanics of deformable bodies. In fact, in the theory of deformable bodies one looses any intrinsic notion of *body frame*, which is the fundamental tool for the description of rigid bodies and their associated

Euler equations. A priori, given any configuration of a non-relativistic continuous body (in particular a N-body system), any barycentric orthogonal frame could be named *body frame* of the system with its associated notion of *vibrations*.

This state of affairs suggested [1] to replace the kinematically accessible region of the non-singular configurations [2] in the (3N-3)-dimensional relative configuration space by a SO(3) principal fiber bundle over a (3N-6)-dimensional base manifold, called *shape* space. The SO(3) fiber on each shape configuration carries the *orientational* variables (e.g. the usual Euler angles) referred to the chosen *body frame*. Then, a local cross section of the principal fiber bundle selects just one orientation of a generic N-body configuration in each fiber (SO(3) orbit). This is in fact equivalent to a *gauge convention* namely, after a preliminary choice of the *shape* variables, to a possible definition of a *body frame* (*reference orientation*). It turns out that this principal bundle is trivial only for N=3, so that in this case global cross sections exist, and in particular the identity cross section may be identified with the *space frame*. Any global cross section is a copy of the 3-body shape space and its coordinatization provides a description of the *internal vibrational* motions associated with the chosen gauge convention for the reference orientation. For $N \geq 4$, however, global cross sections do not exist [4] and the definition of the reference orientation (*body frame*) can be given only locally. This means that the *shape space cannot be identified with a (3N-6)-dimensional sub-manifold* of the (3N-3)-dimensional relative configuration space. The *gauge convention* about the reference orientation and the consequent individuation of the internal *vibrational* degrees of freedom requires the choice of a connection Γ on the SO(3) principal bundle (i.e. a concept of *horizontality*), which leads in turn to the introduction of a SO(3) gauge potential on the base manifold. In this way a natural gauge invariant concept exists of *purely rotational* N-body configurations (*vertical* velocity vector field, i.e. *null shape velocities*). Clearly, a gauge fixing is needed in order to select a particular local cross section and the correlated gauge potential on the shape space. Obviously, *physical quantities like the rotational or vibrational kinetic energies and, in general, any observable feature of the system must be gauge invariant*. On the other hand, in the *orientation-shape* bundle approach, both the space frame and the body frame components of the angular velocity are gauge quantities so that their definition depends upon the gauge convention. See Ref.[1] for a review of the gauge fixings used in molecular physics' literature and, in particular, for the virtues of a special connection C corresponding to the shape configurations with vanishing center-of-mass angular momentum $\vec{S}$. The C connection is defined by introducing local cross sections orthogonal to the fibers with respect to the Riemannian metric dictated by the kinetic energy.

The *orientation-shape* approach replaces the usual Euler kinematics of rigid bodies and implies in general a coupling between the internal *shape* variables and some of the *orientational* degrees of freedom. In Ref.[1] it is interestingly shown that the non-triviality of the SO(3) principal bundle, when extended to continuous deformable bodies, is the core of the physical explanation of the *falling cat* and the *diver*. A characteristic role of SO(3) gauge potentials in this case is to *generate rotations by changing the shape*. This approach is defined in configuration space and leads to *momentum-independent shape variables*, so that it can be easily extended to the quantum level (the N-body Schrödinger equation). In Ref.[1] the Hamiltonian formulation of this framework is also given, but no explicit procedure is worked out for the construction of a canonical Darboux basis for the orientational and shape variables. See Refs.[1, 3] for the existing sets of shape variables for $N = 3, 4$ and for the determination of their physical domain.

Consider now the description of a N-body system in a special relativistic context. Unlike the Newtonian case, manifest Lorentz covariance requires the introduction of 4N degrees of freedom in configuration space and 8N in phase space. In phase space there are N mass-shell first-class constraints ($\epsilon\, p_i^2 - m_i^2 \approx 0$ in the free case [5]). They determine the energies $p_i^o \approx \pm\sqrt{m_i^2 + \vec{p}_i^2}$ and entail that the time variables x_i^o be gauge variables. Such gauge variables can be replaced by a *center-of-mass time* (arbitrariness in the choice of the rate of the center-of-mass clock) and $N-1$ *relative times* (arbitrariness in the synchronization of the clocks associated to each body). The problem of relative times has been a big obstacle to the development of relativistic mechanics (see Refs.[6] for the essential bibliography). The solution of this problem led to the development of parametrized Minkowski theories [6], to the *Wigner-covariant rest-frame instant form of dynamics* referred to the intrinsic inertial rest-frame of the N-body system [6, 7] (see Ref.[8] for Dirac's forms of the dynamics), and to the development of generalized radar coordinates for the synchronization of distant clocks in arbitrary non-inertial frames [9]. A consistent elimination of relative times (i.e. the choice of a convention for the synchronization of clocks, different in general from Einstein's convention), together with a choice of the center-of-mass time, reduce the variables of the N-body system to the same number as in the non-relativistic case. In special relativity the rest-frame instant form of dynamics, which corresponds to Einstein's synchronization convention due to the inertial nature of the rest-frame, implements the *relativistic separation of the center of mass*, so that only $3N-3$ relative coordinates (or $6N-6$ relative phase space variables) survive in the rest-frame.

As it will be shown, this separation is well defined in the rest frame, which is intrinsically defined by the Wigner hyper-planes orthogonal to the conserved 4-momentum of the isolated system. It is important to stress that *each such hyperplane is an instantaneous Euclidean and Wigner-covariant 3-space*. This leads to the characterization of *two* distinct realizations of the Poincare' group, referred to an *abstract observer* sitting outside or inside the Wigner hyper-planes, namely the *external* and the *internal* realization, respectively. Also, we get the characterization of the relevant notions of relativistic external and internal 4- and 3- centers of mass, as well as of a final set of canonical relative variables with respect to the internal canonical 3-center of mass. In absence of interactions, such relative variables are identified by a canonical transformation, which, unlike the non-relativistic case, is *point only in the momenta*. However, in presence of action-at-a-distance interactions among the particles, the canonical transformation identifying the relative variables becomes interaction-dependent. In any case the use of the non-interacting internal 3-centers of mass and relative variables shows that the Wigner hyper-planes constitute the natural framework for the *relativistic theory of orbits*. It is also argued that a future relativistic theory of orbits will be a non-trivial extension of the non-relativistic theory [10], for, in the instant form of relativistic dynamics, the potentials appear both in the Hamiltonian and in the Lorentz boosts.

An important point is that, as shown in Ref.[11], Jacobi normal coordinates, as well as notions like reduced masses and inertia tensors, do not survive in special relativity. Some of such notions can be recovered (and still in a non-unique way) [12] only by replacing the N-body system with a multi-polar expansion. A different strategy must be consequently devised *already at the non-relativistic level* [13].

Due to the presence of the mass-shell first-class constraints, the description of N-body systems in the rest-frame instant form make use of a special class of canonical transformations, of the Shanmugadhasan type [14] and [7]. Such transformations are simultaneously adapted to: i) the Dirac first-class constraints appearing in the Hamiltonian formulation of relativistic models (the transformations have the effect that the constraint equations are replaced by the vanishing of an equal number of new momenta, whose conjugate variables are the Abelianized gauge variables of the system); and to ii) the time-like Poincaré orbits associated with most of their configurations. In the Darboux bases one of the final canonical variables is the square root of the Poincaré invariant P^2 (where P_μ is the conserved time-like four-momentum of the isolated system).

By exploiting the constructive theory of the canonical realizations of Lie groups [15, 16, 17, 18, 19], a new family of canonical transformations was introduced in Ref.[20]. This family of transformations leads to the definition of the so-called *canonical spin bases*, in which also the Pauli-Lubanski Poincaré invariant $W^2 = -P^2\vec{S}_T^2$ for the time-like Poincaré orbits becomes one of the final canonical variables (provided the rest-frame Thomas spin $\vec{S}_T = \sum_{a=1}^{N-1} \vec{S}_a$ is different from zero). The essential point is that the construction of the spin bases *exploits the clustering of spins instead of the Jacobi clustering of centers of mass, which is an ill defined notion at the relativistic level.*

Note that, in spite of its genesis in a relativistic context, the technique used in the determination of the spin bases, related to a *typical form* [15] of the canonical realizations of the E(3) group, *can be easily adapted to the non-relativistic case*, where W^2 is replaced by the invariant $\vec{S}^2$ of the extended Galilei group. The fact that the traditional Jacobi clustering of the centers of mass of the sub-clusters is replaced by the clustering of the spins $\vec{S}_a$, $(a = 1, .., N-1)$ of the sub-clusters, as in the composition of quantum mechanical angular momenta, is *the basic trick that makes the treatment of the non-relativistic N-body problem directly extendible to the relativistic case.* The clustering can be achieved by means of suitable canonical transformations, which *in general are non-point both in the coordinates and the momenta.* This entails that the quantum implementations of such canonical transformations as unitary transformations be non-trivial. The extension of our formalism to quantum mechanics remains an open problem.

Our aim at this point the construction of a canonical Darboux basis adapted to the non-Abelian SO(3) symmetry, both at the non-relativistic and the relativistic level. The three non-Abelian Noether constants of motion $\vec{S} = S^r\, \hat{f}_r$ ($\hat{f}_r$ are the axes of the inertial *laboratory or space frame*) are arranged in these canonical Darboux bases as an array containing the canonical pair S^3, $\beta = tg^{-1}\frac{S^2}{S^1}$ and the unpaired variable $S = |\vec{S}|$ (*scheme A* of the canonical realization of SO(3) [16]; the configurations with $\vec{S} = 0$ are singular and have to be treated separately). The angle canonically conjugated to S, say α, is an *orientational* variable, which, being coupled to the internal *shape* degrees of freedom, cannot be a constant of motion. However, being conjugated to a constant of the motion, it is an *ignorable* variable in the Hamiltonian formalism, so that its equation of motion can be solved by quadratures after the solution of the other equations. In conclusion, in this non-Abelian case one has only two (instead of three as in the Abelian case) commuting constants of motion, namely S and S^3 (like in quantum mechanics). This is also the outcome of the momentum map canonical reduction [21, 22] by means of adapted coordinates. Let us stress that α, S^3, β are a local coordinatization of any co-adjoint orbit of SO(3) contained in the N-body phase space. Each co-adjoint orbit is a 3-dimensional embedded sub-manifold and is endowed with a Poisson structure whose neutral element is α. By fixing non-zero values of the variables S^3, $\beta = tg^{-1}\frac{S^2}{S^1}$ through second-class constraints, one can define a (6N-8)-dimensional reduced phase space. *The canonical reduction cannot be furthered by eliminating S, just because α is* *not a constant of motion*. Yet, the angle α allows us to construct a unit vector $\hat{R}$, orthogonal to $\vec{S}$, such that $\hat{S}$, $\hat{R}$, $\hat{S} \times \hat{R}$ (the notation $\hat{}$ means unit vector) is an orthonormal frame that we call *spin frame*.

The final lacking ingredient for our construction of body frames comes from the group-theoretical treatment of rigid bodies [22] (Chapter IV, Section 10). Such treatment is based on the existence of realizations of the (free and transitive) *left* and *right* Hamiltonian actions of the SO(3) rotation group on either the tangent or cotangent bundle over their configuration space. The generators of the *left* Hamiltonian action [23], which is a *symmetry action*, are the above non-Abelian constants of motion S^1, S^2, S^3, $[\{S^r, S^s\} = \epsilon^{rsu} S^u]$.

At this point let us stress that, in the approach of Ref.[1] the SO(3) principal bundle is built starting from the *relative configuration space* and, upon the choice of a body-frame convention, a gauge-dependent SO(3) *right* action is introduced. The corresponding task in our case is the following: taking into account the *relative phase space* of any isolated system, we have to find out whether one or more SO(3) *right* Hamiltonian actions could be implemented besides the global SO(3) *left* Hamiltonian action. In other words, we have to look for solutions $\check{S}^r$, r=1,2,3, [with $\sum_r (\check{S}^r)^2 = \sum_r (S^r)^2 = S^2$], of the partial differential equations $\{S^r, \check{S}^s\} = 0$, $\{\check{S}^r, \check{S}^s\} = -\epsilon^{rsu} \check{S}^u$ and then build corresponding *left* invariant Hamiltonian vector fields. Alternatively, one may search for the existence of a pair $\check{S}^3$, $\gamma = tg^{-1} \frac{\check{S}^2}{\check{S}^1}$, of canonical variables satisfying $\{\gamma, \check{S}^3\} = -1$, $\{\gamma, S^r\} = \{\check{S}^3, S^r\} = 0$ and also $\{\gamma, \alpha\} = \{\check{S}^3, \alpha\} = 0$. Local theorems given in Refs.[15, 16] guarantee that this is always possible provided $N \geq 3$. Clearly, the functions $\check{S}^r$, which are not constants of the motion, do not generate symmetry actions. What matters here is that each explicitly given right action leads to the characterization of the following two structures: i) a dynamical reference frame (say $\hat{N}$, $\hat{\chi}$, $\hat{N} \times \hat{\chi}$), that we call *dynamical body frame*; ii) a *canonical spin basis* including both the *orientational* and the *shape* variables.

In conclusion, we show that, after the center-of-mass separation, by exploiting the new notions of *dynamical body frames* and *canonical spin bases*, it is possible to build a geometrical and group-theoretical procedure for the common characterization of the rotational kinematics of non-relativistic and relativistic N-body systems. The two cases are treated in Sections II and III, respectively.

In the last Subsection of Section III we show that the relativistic separation of the center of mass, realized by the rest-frame instant form, can be extended to continuous deformable relativistic isolated systems, namely relativistic field configurations, strings and fluids. The action principle of such systems can be transformed into a parametrized Minkowski theory on space-like hyper-surfaces whose embeddings in Minkowski space-time are the *gauge* variables connected with the arbitrariness in the choice of the convention for clock synchronization (namely the *choice of the equal-time Cauchy surfaces* for the field equations). Then the rest-frame instant form on Wigner hyper-planes emerges in a natural way also for fields (ADM canonical metric [24] and tetrad gravity [7] naturally deparametrize to it). All the notions of external and internal 4- and 3-centers of mass can be extended to field configurations under the condition that a collective 4-vector [25], canonically conjugate to the configuration conserved 4-momentum, be definable.

The construction of such a collective variable, with respect to which the energy-momentum distribution of the configuration itself is peaked, is exemplified for a classical real

Klein-Gordon field [26]. New features, like an *internal time variable*, absent in the particle case, emerge for fields and entail that each constant energy surface of the configuration be a disjoint union of symplectic manifolds. These results can be extended to relativistic perfect fluids [27].

The next issue has to do with the simulation of an extended system by means of as few as possible global parameters, sufficient to maintain an acceptable phenomenological description of the system. This can be achieved by replacing the extended system with a multipolar expansion around a world-line describing its mean motion. After many attempts, a general approach to this problem has been given by Dixon in Ref.[28] for special relativity and in Ref.[29] (see also Refs.[30, 31]) for general relativity, after a treatment of the non-relativistic case [29].

With this in view, we show in Section IV that the rest-frame instant-form of the dynamics is the natural framework (extendible to general relativity [7]) for the formulation of relativistic multipolar expansions [12] exhibiting a clear identification of the internal canonical 3-center of mass (with associated spin dipole and higher multi-poles) as a preferred center of motion. Besides a system of N free relativistic particles, we also discuss an *open system* defined by cluster of $n < N$ charged particles inside an isolated system of N charged particles, plus the electro-magnetic field in the radiation gauge [6]. Finally, as a prerequisite to the treatment of relativistic perfect fluids [27], we give some results concerning a configuration of a classical Klein-Gordon field [26], where the collective variable allows identifying a natural center of motion together with the associated Dixon multi-poles.

II. THE NON-RELATIVISTIC CANONICAL SPIN BASES AND DYNAMICAL BODY FRAMES.

Let us consider N free non-relativistic particles of masses m_i, $i = 1, .., N$, described by the configuration variables $\vec{\eta}_i$ and by the momenta $\vec{\kappa}_i$. The Hamiltonian $H = \sum_{i=1}^N \frac{\vec{\kappa}_i^2}{2m_i}$ is restricted by the three first-class constraints $\vec{\kappa}_+ = \sum_{i=1}^N \vec{\kappa}_i \approx 0$ defining the center-of-mass or rest frame.

Let us introduce the following family of point canonical transformations ($m = \sum_{i=1}^N m_i$) realizing the separation of the center of mass from arbitrary relative variables (for instance some set of Jacobi normal coordinates)

$$\boxed{\begin{array}{c} \vec{\eta}_i \\ \hline \vec{\kappa}_i \end{array}} \qquad \longrightarrow \qquad \boxed{\begin{array}{c|c} \vec{q}_{nr} & \vec{\rho}_a \\ \hline \vec{\kappa}_+ & \vec{\pi}_a \end{array}} \tag{2.1}$$

defined by:

$$\vec{\eta}_i \;=\; \vec{q}_{nr} + \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \Gamma_{ai} \vec{\rho}_a, \qquad \vec{\kappa}_i = \frac{m_i}{m} \vec{\kappa}_+ + \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\pi}_a,$$

$$
\begin{aligned}
\vec{q}_{nr} &= \sum_{i=1}^{N} \frac{m_i}{m} \vec{\eta}_i, \qquad \vec{\kappa}_+ = \sum_{i=1}^{N} \vec{\kappa}_i,\\
\vec{\rho}_a &= \sqrt{N} \sum_{i=1}^{N} \gamma_{ai} \vec{\eta}_i, \qquad \vec{\pi}_a = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Gamma_{ai} \vec{\kappa}_i, \qquad \vec{S} = \sum_{a=1}^{N-1} \vec{\rho}_a \times \vec{\pi}_a,
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{ai} &= \gamma_{ai} - \sum_{k=1}^{N} \frac{m_k}{m} \gamma_{ak}, \qquad \gamma_{ai} = \Gamma_{ai} - \frac{1}{N} \sum_{k=1}^{N} \Gamma_{ak},\\
&\sum_{i=1}^{N} \gamma_{ai} = 0, \qquad \sum_{i=1}^{N} \frac{m_i}{m} \Gamma_{ai} = 0,\\
&\sum_{i=1}^{N} \gamma_{ai} \gamma_{bi} = \delta_{ab}, \qquad \sum_{i=1}^{N} \gamma_{ai} \Gamma_{bi} = \delta_{ab},\\
&\sum_{a=1}^{N-1} \gamma_{ai} \gamma_{aj} = \delta_{ij} - \frac{1}{N}, \qquad \sum_{a=1}^{N-1} \gamma_{ai} \Gamma_{aj} = \delta_{ij} - \frac{m_i}{m}.
\end{aligned} \tag{2.2}
$$

Here, the γ_{ai}'s and the Γ_{ai}'s are numerical parameters depending on $\frac{1}{2}(N-1)(N-2)$ free parameters [6].

It can be shown [13] that the relative motion is described by the following Lagrangian and Hamiltonian

$$
\begin{aligned}
L_{rel}(t) &= \frac{1}{2} \sum_{a,b}^{1..N-1} k_{ab}[m_i, \Gamma_{ai}]\, \dot{\vec{\rho}}_a(t) \cdot \dot{\vec{\rho}}_b(t),\\
&\quad k_{ab}[m_i, \Gamma_{ci}] = k_{ba}[m_i, \Gamma_{ci}] = \frac{1}{N} \sum_{i=1}^{N} m_i \Gamma_{ai} \Gamma_{bi},\\
&\quad k^{-1}_{ab}[m_i, \Gamma_{ci}] = N \sum_{i=1}^{N} \frac{\gamma_{ai} \gamma_{bi}}{m_i},\\
&\quad \Downarrow\\
\vec{\pi}_a(t) &= \sum_{b=1}^{N-1} k_{ab}[m_i, \Gamma_{ci}]\, \dot{\vec{\rho}}_b(t),\\
&\Rightarrow \qquad H_{rel} = \frac{1}{2} \sum_{ab}^{1..N-1} k^{-1}_{ab}[m_i, \Gamma_{ai}]\ \vec{\pi}_a(t) \cdot \vec{\pi}_b(t).
\end{aligned} \tag{2.3}
$$

If we add the gauge fixings $\vec{q}_{nr} \approx 0$ and we go to Dirac brackets with respect to the second-class constraints $\vec{\kappa}_+ \approx 0$, $\vec{q}_{nr} \approx 0$, we get a (6N-6)-dimensional reduced phase space spanned by $\vec{\rho}_a$, $\vec{\pi}_a$ and with $\vec{S} \equiv \sum_{a=1}^{N-1} \vec{\rho}_a \times \vec{\pi}_a = \sum_{a=1}^{N-1} \vec{S}_a$. At the non-relativistic level [1] the next problem of the standard approach for each N is the diagonalization of the matrix $k_{ab}[m_i, \Gamma_{ai}]$. The off-diagonal terms of the matrix $k_{ab}[m_i, \Gamma_{ai}]$ are called *mass polarization terms*, while its eigenvalues are the *reduced masses* (see for instance Ref.[32]). In this way the Jacobi normal coordinates $\vec{\rho}_a = \vec{s}_a$, with conjugate momenta $\vec{\pi}_a = \vec{\pi}_{sa}$, are introduced.

In the rest frame the 11 generators of the extended Galilei group (the total mass m is a central charge) [17] are

$$
\begin{aligned}
m &= \sum_{i=1}^N m_i, \qquad E = \sum_{i=1}^N \frac{\vec{\kappa}_i^2}{2m_i} \approx H_{rel}, \qquad \vec{\kappa}_+ = \sum_{i=1}^N \vec{\kappa}_i \approx 0,\\
\vec{J} &= \sum_{i=1}^N \vec{\eta}_i \times \vec{\kappa}_i \approx \vec{S} = \sum_{a=1}^{N-1} \vec{S}_a = \sum_{a=1}^{N-1} \vec{\rho}_a \times \vec{\pi}_a,\\
\vec{K} &= -\sum_{i=1}^N m_i\, \vec{\eta}_i + \vec{\kappa}_+\, t \approx -m\, \vec{q}_{nr},
\end{aligned}
\tag{2.4}
$$

and the gauge fixings $\vec{q}_{nr} \approx 0$ are equivalent to $\vec{K} \approx 0$.

Following the strategy delineated in the Introduction, we search for a geometrical and group-theoretical characterization of further canonical transformations leading to a privileged class of canonical Darboux bases for the N-body. Specifically, they must be adapted to the SO(3) subgroup [15, 16] of the extended Galilei group and contain one of its invariants, namely the modulus of the spin:

1) Every such basis must be a *scheme B* (i.e. a canonical completion of *scheme A*, according to the language of Ref.[15, 16, 17, 20]) for the canonical realization of the rotation group SO(3), viz. it must contain its invariant S and the canonical pair S^3, $\beta = tg^{-1}\frac{S^2}{S^1}$. This entails that, except for α, all the remaining variables in the canonical basis be SO(3) scalars.

2) The existence of the angle α satisfying $\{\alpha, S\} = 1$ and $\{\alpha, S^3\} = \{\alpha, \beta\} = 0$ leads to the geometrical identification of a unit vector $\hat{R}$ orthogonal to $\vec{S}$ and, therefore, of an orthonormal frame, the *spin frame* $\hat{S}$, $\hat{R}$, $\hat{S} \times \hat{R}$.

3) From the equations $\hat{R}^2 = 1$ and $\{\vec{S} \cdot \hat{R}, \hat{R}^i\} = 0$ it follows the symplectic result $\{\hat{R}^i, \hat{R}^j\} = 0$. As a byproduct, we get a canonical realization of a Euclidean group E(3) with generators $\vec{S}$, $\hat{R}$ $[\{\hat{R}^i, \hat{R}^j\} = 0,\ \{\hat{R}^i, S^j\} = \epsilon^{ijk}\hat{R}^k]$ and fixed values of its invariants $\hat{R}^2 = 1$, $\vec{S} \cdot \hat{R} = 0$ (non-irreducible *type 3* realization according to Ref.[20]).

4) In order to implement a SO(3) *Hamiltonian right action* in analogy with the rigid body theory [22], we must construct an orthonormal triad or *body frame* $\hat{N}$, $\hat{\chi}$, $\hat{N} \times \hat{\chi}$. The decomposition

$$
\vec{S} = \check{S}^1 \hat{\chi} + \check{S}^2 \hat{N} \times \hat{\chi} + \check{S}^3 \hat{N} \overset{def}{=} \check{S}^r \hat{e}_r,
\tag{2.5}
$$

identifies the SO(3) scalar generators $\check{S}^r$ of the *right action* provided they satisfy $\{\check{S}^r, \check{S}^s\} = -\epsilon^{rsu}\check{S}^u$. This latter condition together with the obvious requirement that $\hat{N}$, $\hat{\chi}$, $\hat{N} \times \hat{\chi}$ be SO(3) vectors $[\{\hat{N}^r, S^s\} = \epsilon^{rsu}\hat{N}^u$, $\{\hat{\chi}^r, S^s\} = \epsilon^{rsu}\hat{\chi}^u$, $\{\hat{N} \times \hat{\chi}^r, S^s\} = \epsilon^{rsu}\hat{N} \times \hat{\chi}^u]$ entails the equations [33]

$$\{\hat{N}^r, \hat{N}^s\} = \{\hat{N}^r, \hat{\chi}^s\} = \{\hat{\chi}^r, \hat{\chi}^s\} = 0. \tag{2.6}$$

Each solution of these equations identifies a couple of canonical realizations of the E(3) group (non-irreducible, *type 2*): one with generators $\vec{S}$, $\vec{N}$ and non-fixed invariants $\check{S}^3 = \vec{S} \cdot \hat{N}$ and $|\vec{N}|$; another with generators $\vec{S}$, $\vec{\chi}$ and non-fixed invariants $\check{S}^1 = \vec{S} \cdot \hat{\chi}$ and $|\vec{\chi}|$. Such realizations contain the relevant information for constructing the new canonical pair $\check{S}^3$, $\gamma = tg^{-1} \frac{\check{S}^2}{\check{S}^1}$ of SO(3) scalars. Since $\{\alpha, \check{S}^3\} = \{\alpha, \gamma\} = 0$ must hold, it follows [20] that the vector $\hat{N}$ necessarily belongs to the $\hat{S}$-$\hat{R}$ plane. The three canonical pairs S, α, S^3, β, $\check{S}^3$, γ will describe the *orientational* variables of our Darboux basis, while $|\vec{N}|$ and $|\vec{\chi}|$ will belong to the *shape* variables. For each independent right action [i.e. for each solution $\hat{N}$, $\hat{\chi}$ of Eqs.(2.6)], we can identify a *canonical spin basis* containing the above 6 orientational variables and $6N-12$ canonical shape variables. Alternatively, an anholonomic basis can be constructed by replacing the above six variables by $\check{S}^r$ (or S^r) and three uniquely determined Euler angles $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ (see Ref.[13]). Let us stress that the non-conservation of $\check{S}^r$ entails that *the dynamical body frame evolves in a way dictated by the equations of motion*, just as it happens in the rigid body case.

We can conclude that the N-body problem has *hidden structures* leading to the characterization of special *dynamical body frames* which, *being independent of gauge conditions, are endowed with a direct physical meaning.*

A. The 2-body system.

For $N = 2$, a single E(3) group can be defined: it allows the construction of an orthonormal *spin frame* $\hat{S}$, $\hat{R}$, $\hat{R} \times \hat{S}$ in terms of the measurable relative coordinates and momenta of the particles. The relative variables are $\vec{\rho} = \vec{\rho}$, $\vec{\pi}$ and the Hamiltonian is $H_{rel} = \frac{\vec{\pi}^2}{2\mu}$, where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass. The spin is $\vec{S} = \vec{\rho} \times \vec{\pi}$. Let us define the following decomposition

$$\begin{aligned}
\vec{\rho} &= \rho \hat{R}, \qquad \rho = \sqrt{\vec{\rho}^2}, \qquad \hat{R} = \frac{\vec{\rho}}{\rho} = \hat{\rho}, \qquad \hat{R}^2 = 1,\\
\vec{\pi} &= \tilde{\pi} \hat{R} - \frac{S}{\rho} \hat{R} \times \hat{S} = \tilde{\pi} \hat{\rho} - \frac{S}{\rho} \hat{\rho} \times \hat{S},\\
&\tilde{\pi} = \vec{\pi} \cdot \hat{R} = \vec{\pi} \cdot \hat{\rho}, \qquad \hat{S} = \frac{\vec{S}}{S}, \qquad \hat{S} \cdot \hat{R} = 0.
\end{aligned} \tag{2.7}$$

As shown in Ref.[20], it is instrumental considering the following non-point canonical transformation adapted to the SO(3) group, valid when $\vec{S} \neq 0$

$$\begin{array}{|c|}\hline \vec{\rho} \\ \hline \vec{\pi} \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|cc|c|}\hline \alpha & \beta & \rho \\ \hline S & S^3 & \tilde{\pi} \\ \hline \end{array} \quad , \qquad \alpha = tg^{-1} \frac{1}{S} \Big(\vec{\rho} \cdot \vec{\pi} - \frac{(\rho)^2}{\rho^3} \pi^3 \Big). \tag{2.8}$$

In the language of Ref.[20], the two pairs of canonical variables α, S, β, S^3 form the *irreducible kernel* of the *scheme A* of a (non-irreducible, *type 3*,) canonical realization of the group E(3), generated by $\vec{S}$, $\hat{R}$, with fixed values of the invariants $\hat{R}^2 = 1$, $\hat{R} \cdot \vec{S} = 0$, just as the variables S^3, β and S form the *scheme A* of the SO(3) group with invariant S. Geometrically, we have: i) the angle α is the angle between the plane determined by $\vec{S}$ and $\hat{f}_3$ and the plane determined by $\vec{S}$ and $\hat{R}$; ii) the angle β is the angle between the plane $\vec{S}$ - $\hat{f}_3$ and the plane $\hat{f}_3$ - $\hat{f}_1$. Moreover, $S^1 = \sqrt{(S)^2 - (S^3)^2} cos\,\beta$, $S^2 = \sqrt{(S)^2 - (S^3)^2} sin\,\beta$, $\hat{R}^1 = \hat{\rho}^1 = sin\,\beta sin\,\alpha - \frac{S^3}{S} cos\,\beta cos\,\alpha$, $\hat{R}^2 = \hat{\rho}^2 = -cos\,\beta sin\,\alpha - \frac{S^3}{S} sin\,\beta cos\,\alpha$, $\hat{R}^3 = \hat{\rho}^3 = \frac{1}{S}\sqrt{(S)^2 - (S^3)^2} cos\,\alpha$, $\alpha = -tg^{-1}\, \frac{(\hat{S} \times \hat{R})^3}{[\hat{S} \times (\hat{S} \times \hat{R})]^3}$.

In this degenerate case (N=2), the *dynamical* shape variables ρ, $\tilde{\pi}$ coincide with the *static* ones and describe the vibration of the dipole. The rest-frame Hamiltonian for the relative motion becomes ($\check{I}$ is the barycentric tensor of inertia of the dipole) $H_{rel} = \frac{1}{2}\Big[\check{I}^{-1} S^2 + \frac{\tilde{\pi}^2}{\mu}\Big]$, $\check{I} = \mu \rho^2$, while the body frame angular velocity is $\check{\omega} = \frac{\partial H_{rel}}{\partial \check{S}} = \frac{\check{S}}{\check{I}}$.

B. The 3-body system.

For $N = 3$, where we have $\vec{S} = \vec{S}_1 + \vec{S}_2$, a *pair* of E(3) groups emerge, associated with $\vec{S}_1$ and $\vec{S}_2$, respectively. We have now *two* unit vectors $\hat{R}_a$ and *two* E(3) realizations generated by $\vec{S}_a$, $\hat{R}_a$ respectively and fixed invariants $\hat{R}_a^2 = 1$, $\vec{S}_a \cdot \hat{R}_a = 0$ (non-irreducible, type 2, see Ref.[20]). We shall assume $\vec{S} \neq 0$, because the exceptional SO(3) orbit $S = 0$ has to be studied separately by adding $S \approx 0$ as a first-class constraint.

For each value of $a = 1, 2$, we consider the non-point canonical transformation (2.8)

$$
\begin{array}{|c|}\hline \vec{\rho}_a \\ \hline \vec{\pi}_a \\ \hline \end{array}
\quad \longrightarrow \quad
\begin{array}{|cc|c|}\hline \alpha_a & \beta_a & \rho_a \\ \hline S_a & S_a^3 & \tilde{\pi}_a \\ \hline \end{array}\ ,
$$

$$
\begin{aligned}
\vec{\rho}_a &= \rho_a \hat{R}_a, \qquad \rho_a = \sqrt{\vec{\rho}_a^2}, \qquad \hat{R}_a = \frac{\vec{\rho}_a}{\rho_a} = \hat{\rho}_a, \qquad \hat{R}_a^2 = 1, \\
\vec{\pi}_a &= \tilde{\pi}_a \hat{R}_a + \frac{S_a}{\rho_a} \hat{S}_a \times \hat{R}_a, \qquad \tilde{\pi}_a = \vec{\pi}_a \cdot \hat{R}_a.
\end{aligned}
\tag{2.9}
$$

In this case, besides the orthonormal *spin frame*, an orthonormal *dynamical body frame* $\hat{N}$, $\hat{\chi}$, $\hat{N} \times \hat{\chi}$, i.e. a SO(3) Hamiltonian *right* action, can be defined. The *simplest choice*, within the existing arbitrariness [34], for the orthonormal vectors $\vec{N}$ and $\vec{\chi}$ functions only of the relative coordinates is

$$
\begin{aligned}
\vec{N} &= \frac{1}{2}(\hat{R}_1 + \hat{R}_2) = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2), \qquad \hat{N} = \frac{\vec{N}}{|\vec{N}|}, \qquad |\vec{N}| = \sqrt{\frac{1 + \hat{\rho}_1 \cdot \hat{\rho}_2}{2}}, \\
\vec{\chi} &= \frac{1}{2}(\hat{R}_1 - \hat{R}_2) = \frac{1}{2}(\hat{\rho}_1 - \hat{\rho}_2), \qquad \hat{\chi} = \frac{\vec{\chi}}{|\vec{\chi}|}, \qquad |\vec{\chi}| = \sqrt{\frac{1 - \hat{\rho}_1 \cdot \hat{\rho}_2}{2}} = \sqrt{1 - \vec{N}^2}, \\
\vec{N} \times \vec{\chi} &= -\frac{1}{2} \hat{\rho}_1 \times \hat{\rho}_2, \qquad |\vec{N} \times \vec{\chi}| = |\vec{N}||\vec{\chi}| = \frac{1}{2}\sqrt{1 - (\hat{\rho}_1 \cdot \hat{\rho}_2)^2}, \qquad \vec{N} \cdot \vec{\chi} = 0.
\end{aligned}
\tag{2.10}
$$

As said above, (Eq.(2.5), this choice is equivalent to the determination of the non-conserved generators $\check{S}^r$ of a Hamiltonian *right action* of SO(3).

The realization of the E(3) group with generators $\vec{S}$, $\vec{N}$ and non-fixed invariants $\vec{N}^2$, $\vec{S}\cdot\vec{N}$ leads to the final canonical transformation introduced in Ref.[20]

$$
\begin{array}{|c|}
\hline
\vec{\rho}_a \\
\hline
\vec{\pi}_a \\
\hline
\end{array}
\quad \longrightarrow \quad
\begin{array}{|cc|cc|c|}
\hline
\alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \rho_a \\
\hline
S_1 & S_1^3 & S_2 & S_2^3 & \tilde{\pi}_a \\
\hline
\end{array}
\quad \longrightarrow \quad
\begin{array}{|ccc|c|c|}
\hline
\alpha & \beta & \gamma & |\vec{N}| & \rho_a \\
\hline
S = \check{S} & S^3 & \check{S}^3 = \vec{S}\cdot\hat{N} & \xi & \tilde{\pi}_a \\
\hline
\end{array}
$$

$$
\xi \; = \; \frac{\sqrt{2}\sum_{a=1}^{2}(-)^{a+1}\vec{\rho}_a \times \vec{\pi}_a \cdot (\hat{\rho}_2 \times \hat{\rho}_1)}{[1-\hat{\rho}_1\cdot\hat{\rho}_2]\sqrt{1+\hat{\rho}_1\cdot\hat{\rho}_2}}. \tag{2.11}
$$

For N=3 the *dynamical shape variables*, functions of the relative coordinates $\vec{\rho}_a$ only, are $|\vec{N}|$ and ρ_a, while the conjugate shape momenta are ξ, $\tilde{\pi}_a$.

We can now reconstruct $\vec{S}$ and define a *new* unit vector $\hat{R}$ orthogonal to $\vec{S}$ by adopting the prescription given after Eq.(2.8). The vectors $\hat{S}$, $\hat{R}$, $\hat{S}\times\hat{R}$ build up the *spin frame* for N=3. The angle α conjugate to S is explicitly given by $\alpha = -tg^{-1}\frac{(\hat{S}\times\hat{N})^3}{[\hat{S}\times(\hat{S}\times\hat{N})]^3} = -tg^{-1}\frac{(\hat{S}\times\hat{R})^3}{[\hat{S}\times(\hat{S}\times\hat{R})]^3}$. The two expressions of α given here are consistent with the fact that $\hat{S}$, $\hat{R}$ and $\hat{N}$ are coplanar, so that $\hat{R}$ and $\hat{N}$ differ only by a term in $\hat{S}$.

As a consequence of this definition of $\hat{R}$, we get the following expressions for the *dynamical body frame* $\hat{N}$, $\hat{\chi}$, $\hat{N}\times\hat{\chi}$ in terms of the final canonical variables

$$
\begin{aligned}
\hat{N} &= cos\,\psi\hat{S} + sin\,\psi\hat{R} = \frac{\check{S}^3}{S}\hat{S} + \frac{1}{S}\sqrt{(S)^2-(\check{S}^3)^2}\hat{R} = \\
&= \hat{N}[S,\alpha;S^3,\beta;\check{S}^3,\gamma], \\
\\
\hat{\chi} &= sin\,\psi cos\,\gamma\hat{S} - cos\,\psi cos\,\gamma\hat{R} + sin\,\gamma\hat{S}\times\hat{R} = \\
&= \frac{\check{S}^1}{S}\,\hat{S} - \frac{\check{S}^3}{S}\frac{\check{S}^1\,\hat{R} + \check{S}^2\,\hat{S}\times\hat{R}}{\sqrt{(S)^2-(\check{S}^3)^2}} = \hat{\chi}[S,\alpha;S^3,\beta;\check{S}^3,\gamma], \\
\\
& \Downarrow \\
\hat{S} &= sin\,\psi cos\,\gamma\hat{\chi} + sin\,\psi sin\,\gamma\hat{N}\times\hat{\chi} + cos\,\psi\hat{N} \\
&\stackrel{def}{=} \frac{1}{S}\Big[\check{S}^1\hat{\chi} + \check{S}^2\hat{N}\times\hat{\chi} + \check{S}^3\hat{N}\Big], \\
\\
\hat{R} &= -cos\,\psi cos\,\gamma\hat{\chi} - cos\,\psi sin\,\gamma\hat{N}\times\hat{\chi} + sin\,\psi\hat{N}, \\
\\
\hat{R}\times\hat{S} &= -sin\,\gamma\hat{\chi} + cos\,\gamma\hat{N}\times\hat{\chi}.
\end{aligned} \tag{2.12}
$$

While ψ is the angle between $\hat{S}$ and $\hat{N}$, γ is the angle between the plane $\hat{N}-\hat{\chi}$ and the plane $\hat{S}-\hat{N}$. As in the case N=2, α is the angle between the plane $\hat{S}-\hat{f}_3$ and the plane $\hat{S}-\hat{R}$, while β is the angle between the plane $\hat{S}-\hat{f}_3$ and the plane $\hat{f}_3-\hat{f}_1$.

Finally, as shown in Ref.[13], we can perform a sequence of a canonical transformation to Euler angles $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ (explicitly calculable in terms of $\vec{\rho}_a$ and $\vec{\pi}_a$) with their conjugate momenta, followed by a transition to the anholonomic basis used in the orientation-shape bundle approach [1] ($q^\mu = (\rho_1, \rho_2, |\vec{N}|)$, $p_\mu = (\tilde{\pi}_1, \tilde{\pi}_2, \xi)$ are the dynamical shape variables)

$$\begin{array}{|lll|l|l|}\hline \alpha & \beta & \gamma & |\vec{N}| & \rho_a \\ \hline S = \check{S} & S^3 & \check{S}^3 & \xi & \tilde{\pi}_a \\ \hline \end{array} \quad \xrightarrow{non\ can.} \quad \begin{array}{|lll|l|l|}\hline \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} & |\vec{N}| & \rho_a \\ \hline \check{S}^1 & \check{S}^2 & \check{S}^3 & \xi & \tilde{\pi}_a \\ \hline \end{array} \quad = $$

$$\begin{array}{|lll|l|}\hline \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} & q^\mu(\vec{\rho}_a) \\ \hline \check{S}^1 & \check{S}^2 & \check{S}^3 & p_\mu(\vec{\rho}_a, \vec{\pi}_a) \\ \hline \end{array} \quad . \tag{2.13}$$

The Euler angles $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ are determined as the *unique set* of *dynamical orientation variables*.

For $N = 3$ we get a result similar to the orientation-shape bundle approach, namely the following relation between the space and body components of the relative coordinates: $\rho^r_a = \mathcal{R}^r{}_s(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \check{\rho}^s_a(q)$, with $\check{\rho}^1_a(q) = (-)^{a+1} \rho_a \sqrt{1 - \vec{N}^2}$, $\check{\rho}^2_a(q) = 0$, $\check{\rho}^3_a(q) = \rho_a |\vec{N}|$, $S^r = \mathcal{R}^r{}_s(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \check{S}^s$. It can also be shown that for N=3 our definition of *dynamical body frame* can be reinterpreted as a special global cross section (*xxzz gauge*, where x stays for $\hat{\chi}$ and z for $\hat{N}$; an outcome that is independent of the particular choice made for $\vec{N}$ and $\vec{\chi}$) of the trivial SO(3) principal bundle of Ref.[1], namely a privileged choice of body frame. While the above *dynamical body frame* can be identified with the global cross section corresponding to the *xxzz gauge*, all other global cross sections cannot be interpreted as *dynamical body frames* (or *dynamical right* actions), because the SO(3) principal bundle of Ref.[1] is built starting from the relative configuration space and, therefore, it is a *static*, velocity-independent, construction. As a matter of fact, after the choice of the shape configuration variables q^μ and of a space frame in which the relative variables have components ρ^r_a, the approach of Ref.[1] begins with the definition of the body-frame components $\check{\rho}^r_a(q^\mu)$ of the relative coordinates, in the form $\rho^r_a = R^{rs}(\theta^\alpha) \check{\rho}^s_a(q^\mu)$, and then extends it in a *velocity-independent* way to the relative velocities $\dot{\rho}^r_a \stackrel{def}{=} R^{rs}(\theta^\alpha) \check{v}^s_a$. Here R is a rotation matrix, θ^α are arbitrary gauge orientational parameters and $\check{\rho}^r_a(q^\mu)$ is *assumed to depend on the shape variables only.* In our construction we get instead $\rho^r_a = R^{rs}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \check{\rho}^s_a(q^\mu)$ in the *xxzz gauge*, so that in the present case (N=3) all *dynamical* variables of our construction coincide with the *static* variables in the xxzz gauge. On the other hand, in the relative phase space, the construction of the *evolving dynamical body frame* is based on non-point canonical transformation.

C. The N-body system.

For N=4, it holds $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$. Since we have three relative variables $\vec{\rho}_1$, $\vec{\rho}_2$, $\vec{\rho}_3$ and related momenta $\vec{\pi}_1$, $\vec{\pi}_2$, $\vec{\pi}_3$, it is possible to construct *three* sets of *spin frames* and *dynamical body frames* corresponding to the hierarchy of clusterings $((ab)c)$ [i.e. ((12)3), ((23)1), ((31)2); the subscripts a, b, c denote any permutation of $1, 2, 3$] of the relative spins $\vec{S}_a$. The associated three canonical Darboux bases share the three variables S^3, β, S (viz. $\vec{S}$), while *both the remaining three orientational variables and the shape variables depend on the spin clustering.* This entails the existence of three different SO(3) *right* actions with non-conserved canonical generators $\check{S}^r_{(A)}$, A=1,2,3. Consistently, one can define three anholonomic bases $\tilde{\alpha}_{(A)}$, $\tilde{\beta}_{(A)}$, $\tilde{\gamma}_{(A)}$, $\check{S}^r_{(A)}$ and associated shape variables $q^{\mu}_{(A)}$, $p_{(A)\mu}$, $\mu = 1,..,6$, connected by canonical transformations leaving S^r fixed. The relative variables are therefore naturally split in three different ways into 6 dynamical rotational variables and 12 generalized dynamical shape variables. Consequently, we get three possible definitions of *dynamical vibrations.*

By using the explicit construction given in Appendix C of Ref.[13], we define the following sequence of canonical transformations (we assume $S \neq 0$; $S_A \neq 0$, $A = a, b, c$) corresponding to the *spin clustering* pattern $abc \mapsto (ab)c \mapsto ((ab)c)$ [build first the spin cluster (ab), then the spin cluster $((ab)c)$]:

$$
\begin{array}{ll}
& \begin{array}{|c|} \hline \vec{\rho}_a \;\; \vec{\rho}_b \;\; \vec{\rho}_c \\ \vec{\pi}_a \;\; \vec{\pi}_b \;\; \vec{\pi}_c \\ \hline \end{array} \longrightarrow \\[2mm]
\longrightarrow & \begin{array}{|c|c|c|c|} \hline \alpha_a \;\; \beta_a & \alpha_b \;\; \beta_b & \alpha_c \;\; \beta_c & \rho_a \;\; \rho_b \;\; \rho_c \\ \hline S_a \;\; S^3_a & S_b \;\; S^3_b & S_c \;\; S^3_c & \tilde{\pi}_a \;\; \tilde{\pi}_b \;\; \tilde{\pi}_c \\ \hline \end{array} \longrightarrow \\[2mm]
\overset{(ab)c}{\longrightarrow} & \begin{array}{|c|c|c|} \hline \alpha_{(ab)} \;\; \beta_{(ab)} \;\;\;\; \gamma_{(ab)} & \alpha_c \;\; \beta_c & |\vec{N}_{(ab)}| \;\; \rho_a \;\; \rho_b \;\; \rho_c \\ \hline S_{(ab)} \;\; S^3_{(ab)} \;\; \check{S}^3_{(ab)} = \vec{S}_{(ab)} \cdot \hat{N}_{(ab)} & S_c \;\; S^3_c & \xi_{(ab)} \;\; \tilde{\pi}_a \;\; \tilde{\pi}_b \;\; \tilde{\pi}_c \\ \hline \end{array} \longrightarrow \\[2mm]
\longrightarrow & \begin{array}{|c|c|} \hline \alpha_{((ab)c)} \;\; \beta_{((ab)c)} \;\;\;\; \gamma_{((ab)c)} & |\vec{N}_{((ab)c)}| \;\;\;\; \gamma_{(ab)} \;\;\;\; |\vec{N}_{(ab)}| \;\; \rho_a \;\; \rho_b \;\; \rho_c \\ \hline S = \check{S} \;\;\; S^3 \;\;\; \check{S}^3 = \vec{S} \cdot \hat{N}_{((ab)c)} & \xi_{((ab)c)} \;\; \vec{S}_{(ab)} \cdot \hat{N}_{(ab)} \;\; \xi_{(ab)} \;\; \tilde{\pi}_a \;\; \tilde{\pi}_b \;\; \tilde{\pi}_c \\ \hline \end{array} \longrightarrow \\[2mm]
\overset{non\, can.}{\longrightarrow} & \begin{array}{|c|c|} \hline \tilde{\alpha} \;\; \tilde{\beta} \;\; \tilde{\gamma} & |\vec{N}_{((ab)c)}| \;\;\;\; \gamma_{(ab)} \;\;\;\; |\vec{N}_{(ab)}| \;\; \rho_a \;\; \rho_b \;\; \rho_c \\ \hline \check{S}^1 \;\; \check{S}^2 \;\; \check{S}^3 & \xi_{((ab)c)} \;\; \Omega_{(ab)} = \vec{S}_{(ab)} \cdot \hat{N}_{(ab)} \;\; \xi_{(ab)} \;\; \tilde{\pi}_a \;\; \tilde{\pi}_b \;\; \tilde{\pi}_c \\ \hline \end{array} .
\end{array}
\tag{2.14}
$$

The first non-point canonical transformation is based on the existence of the three unit vectors $\hat{R}_A$, $A = a, b, c$, and of three E(3) realizations with generators $\vec{S}_A$, $\hat{R}_A$ and fixed values ($\hat{R}^2_A = 1$, $\vec{S}_A \cdot \hat{R}_A = 0$) of the invariants.

In the next canonical transformation the spins of the *relative particles* a and b are coupled to form the spin cluster (ab), leaving the *relative particle* c as a spectator. We use Eq.(2.10) to define $\vec{N}_{(ab)} = \frac{1}{2}(\hat{R}_a + \hat{R}_b)$, $\vec{\chi}_{(ab)} = \frac{1}{2}(\hat{R}_a - \hat{R}_b)$, $\vec{S}_{(ab)} = \vec{S}_a + \vec{S}_b$, $\vec{W}_{(ab)} = \vec{S}_a - \vec{S}_b$. We get $\vec{N}_{(ab)} \cdot \vec{\chi}_{(ab)} = 0$, $\{N^r_{(ab)}, N^s_{(ab)}\} = \{N^r_{(ab)}, \chi^s_{(ab)}\} = \{\chi^r_{(ab)}, \chi^s_{(ab)}\} = 0$ and a new E(3) realization generated by $\vec{S}_{(ab)}$ and $\vec{N}_{(ab)}$, with non-fixed invariants $|\vec{N}_{(ab)}|$, $\vec{S}_{(ab)} \cdot \hat{N}_{(ab)} \overset{def}{=} \Omega_{(ab)}$. It can be shown that it holds

$$
\begin{aligned}
\vec{\rho}_a &= \rho_a \Big[|\vec{N}_{(ab)}| \hat{N}_{(ab)} + \sqrt{1 - \vec{N}^2_{(ab)}} \hat{\chi}_{(ab)} \Big], \\
\vec{\rho}_b &= \rho_b \Big[|\vec{N}_{(ab)}| \hat{N}_{(ab)} - \sqrt{1 - \vec{N}^2_{(ab)}} \hat{\chi}_{(ab)} \Big], \\
\vec{\rho}_c &= \rho_c \hat{R}_c.
\end{aligned}
\tag{2.15}
$$

It is then possible to define $\alpha_{(ab)}$ and $\beta_{(ab)}$ and a unit vector $\hat{R}_{(ab)}$ with $\vec{S}_{(ab)} \cdot \hat{R}_{(ab)} = 0$, $\{\hat{R}^r_{(ab)}, \hat{R}^s_{(ab)}\} = 0$. This unit vector identifies the *spin cluster* (ab) in the same way as the unit vectors $\hat{R}_A = \hat{\vec{\rho}}_A$ identify the *relative particles* A.

The next step is the coupling of the *spin cluster* (ab) with unit vector $\hat{R}_{(ab)}$ [described by the canonical variables $\alpha_{(ab)}$, $S_{(ab)}$, $\beta_{(ab)}$ $S^3_{(ab)}$] to the *relative particle* c with unit vector $\hat{R}_c$ and described by α_c, S_c, β_c, S^3_c: this characterizes the *spin cluster* $((ab)c)$.

Again, Eq.(2.10) allows to define $\vec{N}_{((ab)c)} = \frac{1}{2}(\hat{R}_{(ab)} + \hat{R}_c)$, $\vec{\chi}_{((ab)c)} = \frac{1}{2}(\hat{R}_{(ab)} - \hat{R}_c)$, $\vec{S} = \vec{S}_{((ab)c)} = \vec{S}_{(ab)} + \vec{S}_c$, $\vec{W}_{((ab)c)} = \vec{S}_{(ab)} - \vec{S}_c$. Since we have $\vec{N}_{((ab)c)} \cdot \vec{\chi}_{((ab)c)} = 0$ and $\{N^r_{((ab)c)}, N^s_{((ab)c)}\} = \{N^r_{((ab)c)}, \chi^s_{((ab)c)}\} = \{\chi^r_{((ab)c)}, \chi^s_{((ab)c)}\} = 0$ due to $\{\hat{R}^r_{(ab)}, \hat{R}^s_c\} = 0$, a new E(3) realization generated by $\vec{S}$ and $\vec{N}_{((ab)c)}$ with non-fixed invariants $|\vec{N}_{((ab)c)}|$, $\vec{S} \cdot \hat{N}_{((ab)c)} = \check{S}^3$ emerges. Then, we can define $\alpha_{((ab)c)}$ and $\beta_{((ab)c)}$ and identify a final unit vector $\hat{R}_{((ab)c)}$ with $\vec{S} \cdot \hat{R}_{((ab)c)} = 0$ and $\{\hat{R}^r_{((ab)c)}, \hat{R}^s_{((ab)c)}\} = 0$.

In conclusion, when $S \neq 0$, we find both a *spin frame* $\hat{S}$, $\hat{R}_{((ab)c)}$, $\hat{R}_{((ab)c)} \times \hat{S}$ and a *dynamical body frame* $\hat{\chi}_{((ab)c)}$, $\hat{N}_{((ab)c)} \times \hat{\chi}_{((ab)c)}$, $\hat{N}_{((ab)c)}$, like in the 3-body case. There is an *important difference*, however: the orthonormal vectors $\vec{N}_{((ab)c)}$ and $\vec{\chi}_{((ab)c)}$ *depend on the momenta* of the relative particles a and b through $\hat{R}_{(ab)}$, so that our results do not share any relation with the N=4 non-trivial SO(3) principal bundle of the orientation-shape bundle approach.

The final 6 *dynamical shape variables* are $q^\mu = \{|\vec{N}_{((ab)c)}|, \gamma_{(ab)}, |\vec{N}_{(ab)}|, \rho_a, \rho_b, \rho_c\}$. While the last four depend only on the original relative coordinates $\vec{\rho}_A$, $A = a, b, c$, the first two depend also on the original momenta $\vec{\pi}_A$: therefore they are *generalized shape variables*. In Ref.[13] it is shown that, instead of $\rho^r_a = R^{rs}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})\check{\rho}^s_a(q^\mu)$, it holds

$$\rho^r_A = \mathcal{R}^{rs}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})\, \check{\rho}^s_A(q^\mu, p_\mu, \check{S}^r), \qquad A = a, b, c. \tag{2.16}$$

Clearly, this result stands completely outside the orientation-shape bundle approach. As a consequence the above anholonomic bases and the associated *evolving dynamical body frames*, however, have no relations with the N=4 *static* non-trivial SO(3) principal bundle of Ref.[1], which admits only local cross sections. Each set of 12 generalized dynamical canonical shape variables is obviously defined modulo canonical transformations so that it should even be possible to find local canonical bases corresponding to the local cross sections of the N=4 *static* non-trivial SO(3) principal bundle of Ref.[1].

Finally, it can be shown that, starting from the Hamiltonian $H_{rel((ab)c)}$ expressed in the final variables, we can define a *rotational Hamiltonian* $H^{(rot)}_{rel((ab)c)}$ and a *vibrational Hamiltonian* $H^{(vib)}_{rel((ab)c)}$ (vanishing of the physical dynamical angular velocity $\check{\omega}^r_{((ab)c)} = 0$), but $H_{rel((ab)c)}$ fails to be the sum of these two Hamiltonians showing once again the non-separability of rotations and vibrations. Let us stress that in the rotational Hamiltonian we find an *inertia-like tensor* depending only on the dynamical shape variables. A similar result, however, does not hold for the spin-angular velocity relation.

The price to be paid for the existence of 3 global *dynamical body frames* for N=4 is a more complicated form of the Hamiltonian kinetic energy. On the other hand, *dynamical vibrations* and *dynamical angular velocity* are measurable quantities in each dynamical body frame.

Our results can be extended to arbitrary N, with $\vec{S} = \sum_{a=1}^{N-1} \vec{S}_a$. There are as many independent ways (say K) of spin clustering patterns as in quantum mechanics. For instance for N=5, $K = 15$: 12 spin clusterings correspond to the pattern $(((ab)c)d)$ and 3 to the pattern $((ab)(cd))$ $[a, b, c, d = 1, .., 4]$. For N=6, $K = 105$: 60 spin clusterings correspond to the pattern $((((ab)c)d)e)$, 15 to the pattern $(((ab)(cd)e)$ and 30 to the pattern $(((ab)c)(de))$ $[a, b, c, d, e = 1, .., 5]$. Each spin clustering is associated to: a) a related *spin frame*; b) a related *dynamical body frame*; c) a related Darboux spin canonical basis with orientational variables S^3, β, S, $\alpha_{(A)}$, $\check{S}^3_{(A)}$, $\gamma_{(A)} = tg^{-1}\, \frac{\check{S}^2_{(A)}}{\check{S}^1_{(A)}}$, $A = 1, .., K$ (their anholonomic counterparts are $\tilde{\alpha}_{(A)}$, $\tilde{\beta}_{(A)}$, $\tilde{\gamma}_{(A)}$, $\check{S}^r_{(A)}$ with uniquely determined orientation angles) and shape variables $q^\mu_{(A)}$, $p_{\mu(A)}$, $\mu = 1, .., 3N-6$. Furthermore, for $N \geq 4$ we find the following relation between spin and angular velocity: $\check{S}^r = \mathcal{I}^{rs}(q^\mu_{(A)})\, \check{\omega}^s_{(A)} + f^\mu(q^\nu_{(A)}) p_{(A)\mu}$.

Therefore, for $N \geq 4$ our sequence of canonical and non-canonical transformations leads to the following result, to be compared with Eq.(2.13) of the 3-body case

$$
\begin{array}{|c|} \hline \vec{\rho}_A \\ \hline \vec{\pi}_A \\ \hline \end{array}
\quad \xrightarrow{non\ can.} \quad
\begin{array}{|ccc|c|} \hline \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} & q^\mu(\vec{\rho}_A, \vec{\pi}_A) \\ \hline \check{S}^1 & \check{S}^2 & \check{S}^3 & p_\mu(\vec{\rho}_A, \vec{\pi}_A) \\ \hline \end{array} \quad . \tag{2.17}
$$

Therefore, for $N \geq 4$ and with $S \neq 0$, $S_A \neq 0$, $A = a, b, c$, namely when the standard (3N-3)-dimensional orientation-shape bundle is not trivial, the original (6N-6)-dimensional relative phase space admits as many *dynamical body frames* as spin canonical bases, which are globally defined (apart isolated coordinate singularities) for the non-singular N-body configurations with $\vec{S} \neq 0$ (and with non-zero spin for each spin sub-cluster). These *dynamical body frames* do not correspond to local cross sections of the static non-trivial orientation-shape SO(3) principal bundle and the *spin canonical bases* do not coincide with the canonical bases associated to the static theory.

The $\vec{S} = 0$, *C-horizontal*, cross section of the *static* SO(3) principal bundle corresponds to N-body configurations that cannot be included in the previous Hamiltonian construction based on the canonical realizations of SO(3): these configurations (which include the singular ones) have to be analyzed independently since they are related to the exceptional orbit of SO(3), whose little group is the whole group.

While physical observables must be obviously independent of the gauge-dependent *static body frames*, they do depend on the *dynamical body frame*, whose axes are operationally defined in terms of the relative coordinates and momenta of the particles. In particular, a *dynamical* definition of *vibration*, which replaces the $\vec{S} = 0$, *C-horizontal*, cross section of the *static* approach [1], is based on the requirement that the components of the *angular velocity* vanish. Actually, the angular velocities with respect to the dynamical body frames become now *measurable quantities*, in agreement with the phenomenology of extended deformable bodies (see, e.g., the treatment of spinning stars in astrophysics).

In this connection, let us recall that the main efforts done in developing canonical transformations for the N-body problem have been done in the context of celestial mechanics. As a consequence, these transformations are necessarily adapted to the Newtonian gravitational potential. As well known, the Kepler problem and the harmonic potential are the only interactions admitting extra dynamical symmetries besides the rotational one. The particularity of the final canonical transformations which have been worked out in that mechanical context is that the Hamiltonian figures as one of the final momenta. This is in particular true for the N=2 body problem (Delaunay variables; our β is the *longitude of the ascending node* Ω [10]) and for the N=3 body problem (Jacobi's method of *the elimination of the nodes*).

On the other hand, it is well-known that, for generic non-integrable interactions, putting the Hamiltonian in the canonical bases is quite useless, since it does not bring to *isolating integrals* of the motion having the capability of reducing the dimensionality of phase space. Here we stress again that our construction rests only on the left and right actions of the SO(3) group and is therefore completely *independent of form of the interactions*.

The expression of the relative Hamiltonian H_{rel} for $N = 3$ in these variables is a complicated function, given in Ref.[13], which fails to be the sum of a rotational plus a vibrational part. It is an open problem for N=3 and $N \geq 4$ to check whether suitable choices of the numerical constants γ_{ai}, more general body frames obtained by exploiting the freedom of making arbitrary configuration-dependent rotations [34] and/or suitable canonical transformations of the shape variables among themselves can simplify the free Hamiltonian and/or some type of interaction.

In conclusion, for each N, we get a finite number of physically well-defined separations between *rotational* and *vibrational* degrees of freedom. The unique *body frame* of rigid bodies is replaced here by a discrete number of *evolving dynamical body frames* and of *spin canonical bases*. Both of them are grounded on patterns of spin couplings which are the direct analogues of the coupling of quantum-mechanical angular momenta.

III. THE RELATIVISTIC CENTER-OF-MASS PROBLEM, RELATIVE VARIABLES AND THE ROTATIONAL KINEMATICS IN SPECIAL RELATIVITY.

Let us see what happens when we replace Galilean space plus time with Minkowski space-time, already for the simple model-system of N free scalar positive-energy particles.

First of all we have to describe a relativistic scalar particle. Among the various possibilities (see Refs.[6, 35] for a review of the various options) we will choose the manifestly Lorentz covariant approach based on Dirac's first-class constraints

$$p_i^2 - \epsilon m_i^2 \approx 0. \tag{3.1}$$

The associated Lagrangian description is based on the 4-vector positions $x_i^\mu(\tau)$ and the action $S = \int d\tau \Big(- \epsilon \sum_i m_i \sqrt{\epsilon \dot{x}_i^2(\tau)}\Big)$, where τ is a Lorentz scalar *mathematical* time, i.e. an affine parameter for the particle time-like world-lines. Then, Lorentz covariance implies singular Lagrangians and the associated Dirac's theory of constraints for the Hamiltonian description. The individual time variables $x_i^o(\tau)$ are the *gauge variables* associated to the mass-shell constraints, which have the two topologically disjoint solutions $p_i^o \approx \pm \sqrt{m_i^2 + \vec{p}_i^2}$. As discussed in Ref.[36] and [6] this implies that:

i) a combination of the time variables can be identified with the clock of one arbitrary observer labeling the evolution of the isolated system;

ii) the $N-1$ relative times are related to observer's freedom of looking at the N particles either at the same time or with any prescribed relative delay, or, in other words, to the convention for synchronization of distant clocks (definition of equal-time Cauchy surfaces on which particle's clocks are synchronized) used by the observer to characterize the temporal evolution of the particles [9].

Introducing interactions in this picture without destroying the first-class nature of the constraints is a well-known difficult problem, reviewed in Refs.[6, 35], where also the models with second-class constraints are considered and compared.

A. Parametrized Minkowski theories.

If the particle is charged and interacts with a dynamical electromagnetic field, a *problem of covariance* appears. The standard description is based on the action

$$S=-\epsilon m\int d\tau\sqrt{\epsilon\dot{x}^2(\tau)}-e\int d\tau\int d^4z\delta^4(z-x(\tau))\dot{x}^\mu(\tau)A_\mu(z)-\frac{1}{4}\int d^4zF^{\mu\nu}(z)F_{\mu\nu}(z). \tag{3.2}$$

By evaluating the canonical momenta of the isolated system, *charged particle plus electromagnetic field*, we find two primary constraints:

$$\chi(\tau)=\Big(p-eA(x(\tau))\Big)^2-\epsilon m^2\approx 0,\qquad \pi^o(z^o,\vec{z})\approx 0. \tag{3.3}$$

It is immediately seen that, since there is no concept of *equal time*, it is impossible to evaluate the Poisson bracket of these constraints. Also, due to the same reason, the Dirac Hamiltonian, which would be $H_D=H_c+\lambda(\tau)\chi(\tau)+\int d^3z\lambda^o(z^o,\vec{z})\pi^o(z^o,\vec{z})$ with H_c the canonical Hamiltonian and with $\lambda(\tau)$, $\lambda^o(z^o,\vec{z})$ Dirac's multipliers, does not make sense. This problem is present even at the level of the Euler-Lagrange equations, specifically in the formulation of a *Cauchy problem* for a system of coupled equations some of which are ordinary differential equations in the affine parameter τ along the particle world-line, while the others are partial differential equations depending on Minkowski coordinates z^μ. Since the problem is due the lack of a covariant concept of *equal time* between field and particle variables, a new formulation is needed.

In Ref.[6], after a discussion of the many-time formalism, a solution of the problem was found within a context suited to incorporate the gravitational field. The starting point is an arbitrary 3+1 splitting of Minkowski space-time with space-like hyper-surfaces (see Ref.[9] for more details on the admissible splittings). After choosing the world-line of an arbitrary observer, a centroid $x^\mu_s(\tau)$, as origin, a set of generalized radar 4-coordinates [9], adapted to the splitting, is provided by the τ affine parameter of the centroid world-line together with a system of curvilinear 3-coordinates $\vec{\sigma}\stackrel{def}{=}\{\sigma^r\}$ vanishing on the world-line. In this way we get an arbitrary extended non-inertial frame centered on the (in general accelerated) observer described by the centroid. The coordinates $(\tau,\vec{\sigma})$ are generalized radar coordinates depending upon the choice of the centroid and the splitting. The space-like hyper-surfaces are described by their embedding $z^\mu(\tau,\vec{\sigma})$ in Minkowski space-time. The metric induced by the change of coordinates $x^\mu\mapsto\sigma^A=(\tau,\vec{\sigma})$ is $g_{AB}(\tau,\vec{\sigma})=\partial_A z^\mu(\tau,\vec{\sigma})\,\eta_{\mu\nu}\,\partial_B z^\nu(\tau,\vec{\sigma})$ ($\partial_A z^\mu(\tau,\vec{\sigma})=\partial z^\mu(\tau,\vec{\sigma})/\partial\sigma^A$). This is essentially Dirac's reformulation [37] of classical field theory (suitably extended to particles) on arbitrary space-like hyper-surfaces (*equal time* or simultaneity Cauchy surfaces). Note, incidentally, that it is also the classical basis of the Tomonaga-Schwinger formulation of quantum field theory.

Given any isolated system, containing any combination of particles, strings and fields and described by an action principle, one is lead to a reformulation of it as a *parametrized Minkowski theory* [6], with the extra bonus that the theory is already prepared for the coupling to gravity in its ADM formulation (Ref.[24]). This is done by coupling its action to an external gravitational field $g_{\mu\nu}(x)$ and then by replacing the external 4-metric with $g_{AB}(\tau,\vec{\sigma})$. In this way we get an action depending on the isolated system and on the embedding $z^\mu(\tau,\vec{\sigma})$ as configurational variables. Since the action is invariant under separate

τ-reparametrizations and space-diffeomorphisms (frame-preserving diffeomorphisms [9]), additional first-class constraints are needed to ensure the independence of the description from the choice of the 3+1 splitting, namely from the convention chosen for the synchronization of distant clocks identifying the instantaneous 3-space to be used as Cauchy surface. The embedding configuration variables $z^\mu(\tau, \vec{\sigma})$ are the *gauge* variables associated with this kind of special-relativistic general covariance and describe all the possible *inertial effects* compatible with special relativity.

Let us come back to the discussion of free particles within the parametrized Minkowski theory approach. Since the intersection of a time-like world-line with a space-like hypersurface, corresponding to a value τ of the time parameter, is identified by 3 numbers $\vec{\sigma} = \vec{\eta}(\tau)$ and *not by four*, each particle must have a well-defined sign of the energy. We cannot describe, therefore, the two topologically disjoint branches of the mass hyperboloid simultaneously as in the standard manifestly Lorentz-covariant approach. Then, there are no more mass-shell constraints. Each particle with a definite sign of the energy is described by the canonical coordinates $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$ (like in non-relativistic physics), while the 4-position of the particles is given by $x_i^\mu(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau))$. The 4-momenta $p_i^\mu(\tau)$ are $\vec{\kappa}_i$-dependent solutions of $p_i^2 - \epsilon m_i^2 = 0$ with the given sign of the energy.

The system of N free scalar and positive energy particles is described by the action [6]

$$
\begin{aligned}
S \,&=\, \int d\tau d^3\sigma\, \mathcal{L}(\tau, \vec{\sigma}) = \int d\tau L(\tau),\\
&\mathcal{L}(\tau, \vec{\sigma}) = - \sum_{i=1}^N \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) m_i \sqrt{g_{\tau\tau}(\tau, \vec{\sigma}) + 2 g_{\tau\check{r}}(\tau, \vec{\sigma}) \dot{\eta}_i^{\check{r}}(\tau) + g_{\check{r}\check{s}}(\tau, \vec{\sigma}) \dot{\eta}_i^{\check{r}}(\tau) \dot{\eta}_i^{\check{s}}(\tau)},\\
&L(\tau) = - \sum_{i=1}^N m_i \sqrt{g_{\tau\tau}(\tau, \vec{\eta}_i(\tau)) + 2 g_{\tau\check{r}}(\tau, \vec{\eta}_i(\tau)) \dot{\eta}_i^{\check{r}}(\tau) + g_{\check{r}\check{s}}(\tau, \vec{\eta}_i(\tau)) \dot{\eta}_i^{\check{r}}(\tau) \dot{\eta}_i^{\check{s}}(\tau)},
\end{aligned} \tag{3.4}
$$

where the configuration variables are $z^\mu(\tau, \vec{\sigma})$ and $\vec{\eta}_i(\tau)$, i=1,..,N. As said the action is invariant under frame-dependent diffeomorphisms [9]. In phase space the Dirac Hamiltonian is a linear combination of the following first-class constraints in strong involution

$$
\begin{aligned}
\mathcal{H}_\mu(\tau, \vec{\sigma}) \,&=\, \rho_\mu(\tau, \vec{\sigma}) - l_\mu(\tau, \vec{\sigma}) \sum_{i=1}^N \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \sqrt{m_i^2 - \gamma^{\check{r}\check{s}}(\tau, \vec{\sigma}) \kappa_{i\check{r}}(\tau) \kappa_{i\check{s}}(\tau)} - \\
&-\, z_{\check{r}\mu}(\tau, \vec{\sigma}) \gamma^{\check{r}\check{s}}(\tau, \vec{\sigma}) \sum_{i=1}^N \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \kappa_{i\check{s}} \approx 0.
\end{aligned} \tag{3.5}
$$

The conserved Poincaré generators are

$$
p_s^\mu = \int d^3\sigma \rho^\mu(\tau, \vec{\sigma}), \qquad J_s^{\mu\nu} = \int d^3\sigma [z^\mu(\tau, \vec{\sigma}) \rho^\nu(\tau, \vec{\sigma}) - z^\nu(\tau, \vec{\sigma}) \rho^\mu(\tau, \vec{\sigma})]. \tag{3.6}
$$

B. The rest-frame instant form on the Wigner hyper-planes.

Due to the special-relativistic general covariance implying the gauge equivalence of the descriptions associated to different admissible 3+1 splitting of Minkowski space-time in parametrized theories, the foliation can be restricted to space-like hyper-planes. In particular, for each configuration of the isolated system with time-like 4-momentum, the leaves are best chosen as the hyper-planes orthogonal to the conserved total 4-momentum (*Wigner hyper-planes*). Note that this special foliation is *intrinsically* determined by the configuration of the isolated system alone. This leads to the definition of the *Wigner-covariant rest-frame instant form of dynamics* [6], for every isolated system whose configurations have well-defined and finite Poincaré generators with time-like total 4-momentum [8]. An inertial rest frame for the system is obtained by restricting the centroid world-line to a straight line orthogonal to Wigner hyper-planes.

On a Wigner hyperplane, we obtain the following constraints [the only remnants of the constraints (3.5)] and the following Dirac Hamiltonian [6]

$$
\begin{aligned}
&\epsilon_s - M_{sys} \approx 0, \qquad \vec{\kappa}_+ = \sum_{i=1}^N \vec{\kappa}_i \approx 0, \qquad M_{sys} = \sum_{i=1}^N \sqrt{m_i^2 + \vec{\kappa}_i^2},\\
H_D &= \lambda(\tau)[\epsilon_s - M_{sys}] - \vec{\lambda}(\tau) \sum_{i=1}^N \vec{\kappa}_i,\\
\dot{x}^\mu_s(\tau) &\approx -\lambda(\tau) u^\mu(p_s) + \epsilon^\mu_r(u(p_s))\lambda_r(\tau), \qquad \dot{\tilde{x}}^\mu_s(\tau) = -\lambda(\tau) u^\mu(p_s),\\
x^\mu_s(\tau) &= x^\mu_o + u^\mu(p_s)\, T_s + \epsilon^\mu_r(u(p_s)) \int_o^\tau d\tau_1\, \lambda_r(\tau_1).
\end{aligned} \tag{3.7}
$$

The embedding describing Wigner hyper-planes is $z^\mu(\tau, \vec{\sigma}) = x^\mu_s(\tau) + \epsilon^\mu_r(u(p_s))\sigma^r$, where $\epsilon^\mu_o(u(p)) = u^\mu(p) = p^\mu/\sqrt{\epsilon\, p^2}$ and $\epsilon^\mu_r(u(p)) = \Big(- u_r(p); \delta^i_r - \frac{u^i(p)\, u_r(p)}{1+u^o(p)}\Big)$ are the columns of the standard Wigner boost, $\epsilon^\mu_\nu(u(p_s)) = L^\mu{}_\nu(p_s, \overset{\circ}{p}_s)$, connecting the time-like 4-vector p^μ to its rest-frame form, $\overset{\circ}{p} = \sqrt{\epsilon\, p^2}\,(1; \vec{0})$. As a consequence the space indices "r" now transform as Wigner spin-1 3-vectors.

The 3 Dirac's multipliers $\vec{\lambda}(\tau)$ describe the *classical zitterbewegung* of the centroid $x^\mu_s(\tau)$: each gauge-fixing $\vec{\chi}(\tau) \approx 0$ to the three first-class constraints $\vec{\kappa}_+ \approx 0$ gives a different determination of the multipliers $\vec{\lambda}(\tau)$ and therefore identifies a different world-line $x^{(\vec{\chi})\mu}_s(\tau)$ for the covariant non-canonical centroid.

The various spin tensors and vectors evaluated with respect to the centroid are [6]

$$
\begin{aligned}
J^{\mu\nu}_s &= x^\mu_s p^\nu_s - x^\nu_s p^\mu_s + S^{\mu\nu}_s,\\
S^{\mu\nu}_s &= [u^\mu(p_s)\epsilon^\nu(u(p_s)) - u^\nu(p_s)\epsilon^\mu(u(p_s))]\bar{S}^{\tau r}_s + \epsilon^\mu(u(p_s))\epsilon^\nu(u(p_s))\bar{S}^{rs}_s,
\end{aligned}
$$

$$\bar{S}_s^{AB} \;=\; \epsilon^A_\mu(u(p_s))\epsilon^B_\nu(u(p_s))S_s^{\mu\nu} = \Big(\bar{S}_s^{rs} \equiv \sum_{i=1}^N(\eta_i^r\kappa_i^s - \eta_i^s\kappa_i^r);\quad \bar{S}_s^{\tau r} \equiv -\sum_{i=1}^N \eta_i^r\sqrt{m_i^2c^2 + \vec{\kappa}_i^2}\Big),$$

$$\vec{\bar{S}} \;\equiv\; \vec{\bar{S}} = \sum_{i+1}^N \vec{\eta}_i \times \vec{\kappa}_i \approx \sum_{i=1}^N \vec{\eta}_i \times \vec{\kappa}_i - \vec{\eta}_+ \times \vec{\kappa}_+ = \sum_{a=1}^{N-1}\vec{\rho}_a \times \vec{\pi}_a. \tag{3.8}$$

Note that $L_s^{\mu\nu} = x_s^\mu\, p_s^\nu - x_s^\nu\, p_s^\mu$ and $S_s^{\mu\nu}$ are not constants of the motion due to the *classical zitterbewegung*.

Let us now summarize some relevant points of the rest-frame instant form on Wigner hyper-planes, since this formulation clarifies the role of the various relativistic centers of mass. This is a long standing problem which arose just after the foundation of special relativity in the first decade of the last century. It became soon clear that the problem of the relativistic center of mass is highly non-trivial: no definition can enjoy all the properties of the ordinary non-relativistic center of mass. See Refs.[38, 39, 40, 41, 42, 43] for a partial bibliography of all the existing attempts and Ref.[44] for reviews.

Let us stress that, as said in the Introduction, *our approach leads to a doubling of the usual concepts*: there is an *external* viewpoint (see Subsection C) and an *internal* viewpoint (see Subsection D) with respect to the Wigner hyper-planes. While in Newton physics for any given absolute time there exists an absolute instantaneous 3-space and a unique center of mass with a unique associated 3-momentum $\vec{P}$ (vanishing in the rest frame), in special relativity the notion of instantaneous 3-space requires a convention for the synchronization of distant clocks. In other words we are forced: i) to introduce a 3+1 splitting of Minkowski space-time and an arbitrary accelerated observer; ii) for the sake of simplicity to restrict ourselves to inertial frames, namely to an inertial observer with a 4-momentum p_s^μ and to foliations with space-like hyper-planes orthogonal to this 4-momentum (Wigner hyper-planes); iii) to describe the isolated system inside each of these instantaneous rest-frame 3-spaces, where its total 3-momentum vanishes. As a consequence, the unique non-relativistic center-of-mass 3-momentum $\vec{P}$, vanishing in the rest frame, is replaced by two *independent* notions: i) an *external* 4-momentum $p_s^\mu = (p_s^o, \vec{p}_s = \epsilon_s\vec{k}_s)$ describing the orientation of the instantaneous 3-space with respect to an arbitrary reference inertial observer; ii) an *internal* 3-momentum inside the instantaneous 3-space, which vanishes, $\vec{\kappa}_+ \approx 0$, as a definition of rest frame. The rest-frame instant form of dynamics is the natural framework to visualize this doubling.

In the rest-frame instant form only four first-class constraints survive so that the original configurational variables $z^\mu(\tau, \vec{\sigma})$, $\vec{\eta}_i(\tau)$ and their conjugate momenta $\rho_\mu(\tau, \vec{\sigma})$, $\vec{\kappa}_i(\tau)$ are reduced to:

i) a decoupled point $\tilde{x}_s^\mu(\tau)$, p_s^μ (the only remnant of the space-like hyper-surface) with a positive mass $\epsilon_s = \sqrt{\epsilon p_s^2}$ determined by the first-class constraint $\epsilon_s - M_{sys} \approx 0$ (M_{sys} is the invariant mass of the isolated system). Its rest-frame Lorentz scalar time $T_s = \frac{\tilde{x}_s \cdot p_s}{\epsilon_s}$ is put equal to the mathematical time as a gauge fixing $T_s - \tau \approx 0$ to the previous constraint. The unit time-like 4-vector $u^\mu(p_s) = p_s^\mu/\epsilon_s$ is orthogonal to the Wigner hyper-planes and describes their orientation in the chosen inertial frame.

Here, $\tilde{x}_s^\mu(\tau)$ is a *non-covariant canonical* variable describing the decoupled canonical *external 4-center of mass*. It plays the role of a kinematical external 4-center of mass and of a decoupled observer with his parametrized clock (*point particle clock*). Its velocity $\dot{\tilde{x}}^\mu_s(\tau)$ is parallel to p^μ_s, so that it has no *classical zitterbewegung*. The connection between the centroid $x^\mu_s(\tau) = z^\mu(\tau, \vec{0})$ and $\tilde{x}^\mu_s(\tau)$ and the associated decomposition of the angular momentum are

$$
\begin{aligned}
\tilde{x}^\mu_s(\tau) &\stackrel{def}{=} z^\mu(\tau, \vec{\tilde{\sigma}}) = x^\mu_s(\tau) - \frac{1}{\epsilon_s(p^o_s + \epsilon_s)} \Big[p_{s\nu} S^{\nu\mu}_s + \epsilon_s (S^{o\mu}_s - S^{o\nu}_s \frac{p_{s\nu} p^\mu_s}{\epsilon^2_s}) \Big], \\
J^{\mu\nu}_s &= \tilde{x}^\mu_s p^\nu_s - \tilde{x}^\nu_s p^\mu_s + \tilde{S}^{\mu\nu}_s, \\
\tilde{S}^{\mu\nu}_s &= S^{\mu\nu}_s + \frac{1}{\sqrt{\epsilon p^2_s}(p^o_s + \sqrt{\epsilon p^2_s})} \Big[p_{s\beta}(S^{\beta\mu}_s p^\nu_s - S^{\beta\nu}_s p^\mu_s) + \sqrt{p^2_s}(S^{o\mu}_s p^\nu_s - S^{o\nu}_s p^\mu_s) \Big], \\
&\qquad \tilde{S}^{ij}_s = \delta^{ir}\delta^{js}\bar{S}^{rs}_s, \qquad \tilde{S}^{oi}_s = - \frac{\delta^{ir}\,\bar{S}^{rs}_s\, p^s_s}{p^o_s + \sqrt{\epsilon p^2_s}},
\end{aligned}
\tag{3.9}
$$

Now both $\tilde{L}^{\mu\nu}_s = \tilde{x}^\mu_s\, p^\nu_s - \tilde{x}^\nu_s\, p^\mu_s$ and $\tilde{S}^{\mu\nu}_s$ are conserved.

ii) the $6N$ particle canonical variables $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$ *inside* the Wigner hyper-planes. They are restricted by the three first-class constraints (the *rest-frame conditions*) $\vec{\kappa}_+ = \sum_{i=1}^N \vec{\kappa}_i \approx 0$. They are Wigner spin-1 3-vectors, like the coordinates $\vec{\sigma}$.

The canonical variables $\tilde{x}^\mu_s$, p^μ_s for the *external* 4-center of mass, can be replaced by the canonical pairs [45]

$$
T_s = \frac{p_s \cdot \tilde{x}_s}{\epsilon_s} = \frac{p_s \cdot x_s}{\epsilon_s}, \qquad \epsilon_s = \pm\sqrt{\epsilon p^2_s}, \qquad \vec{z}_s = \epsilon_s (\vec{\tilde{x}}_s - \frac{\vec{p}_s}{p^o_s} \tilde{x}^o_s), \qquad \vec{k}_s = \frac{\vec{p}_s}{\epsilon_s}. \tag{3.10}
$$

In the rest-frame instant form, this non-point canonical transformation can be summarized as

$$
\begin{array}{|c|c|}
\hline
\tilde{x}^\mu_s & \vec{\eta}_i \\
\hline
p^\mu_s & \vec{\kappa}_i \\
\hline
\end{array}
\qquad \longrightarrow \qquad
\begin{array}{|c|c|c|}
\hline
\epsilon_s & \vec{z}_s & \vec{\eta}_i \\
\hline
T_s & \vec{k}_s & \vec{\kappa}_i \\
\hline
\end{array}
\tag{3.11}
$$

The addition of the gauge-fixing $T_s - \tau \approx 0$ for the first-class constraint $\epsilon_s - M_{sys} \approx 0$, building a pair of second-class constraints and implying $\lambda(\tau) = -1$ (due to the explicit τ-dependence of the gauge fixing), leads to the elimination of T_s and ϵ_s. We are then left with a decoupled free point (*point particle clock*) of mass M_{sys} and canonical 3-coordinates $\vec{z}_s$, $\vec{k}_s$. The position $\vec{q}_s = \vec{z}_s/\epsilon_s$ is *the classical analogue of the Newton-Wigner 3-position operator* [39] and shares with it the reduced covariance under the Euclidean subgroup of the Poincaré group.

The invariant mass M_{sys} of the system, which is also its *internal* energy, replaces the non-relativistic Hamiltonian H_{rel} for the relative degrees of freedom. For $c \rightarrow \infty$ we have $M_{sys} - m\, c^2 \rightarrow H_{rel}$. M_{sys} generates the evolution in the rest-frame Lorentz scalar time T_s in the rest frame: in this way we get the same equations of motion as before the addition of the gauge fixing. This reminds of the frozen Hamilton-Jacobi theory, in which the time evolution can be reintroduced by using the energy generator of the Poincaré group as Hamiltonian. As a consequence, after the gauge fixing $T_s - \tau \approx 0$, the final Hamiltonian and the embedding of the Wigner hyperplane into Minkowski space-time become

$$
\begin{aligned}
H_D &= M_{sys} - \vec{\lambda}(\tau) \cdot \vec{\kappa}_+, \\
z^\mu(\tau, \vec{\sigma}) &= x^\mu_s(\tau) + \epsilon^\mu_r(u(p_s))\sigma^r = x^\mu_s(0) + u^\mu(p_s)\tau + \epsilon^\mu_r(u(p_s))\sigma^r, \\
with \qquad & \dot{x}^\mu_s(\tau) \stackrel{\circ}{=} \frac{d\, x^\mu_s(\tau)}{d\tau} + \{x^\mu_s(\tau), H_D\} = u^\mu(p_s) + \epsilon^\mu_r(u(p_s))\lambda_r(\tau),
\end{aligned} \tag{3.12}
$$

where $x^\mu_s(0)$ is an arbitrary point. This equation visualizes the role of the Dirac multipliers as sources of the *classical zittebewegung* and consequently the *gauge nature* of this latter. Let us remark that the constant $x^\mu_s(0)$ [and $\tilde{x}^\mu_s(0)$] is arbitrary, reflecting the arbitrariness in the absolute location of the origin of the *internal* coordinates on each hyperplane in Minkowski space-time.

Inside the Wigner hyperplane three degrees of freedom of the isolated system, describing an *internal* center-of-mass 3-variable $\vec{\sigma}_{com}$ conjugate to $\vec{\kappa}_+$ (when the $\vec{\sigma}_{com}$ are canonical variables they are denoted $\vec{q}_+$) are *gauge* variables. The natural gauge fixing in order to eliminate the three first-class constraints $\vec{\kappa}_+ \approx 0$ is $\vec{\chi} = \vec{q}_+ \approx 0$ which implies $\lambda_r(\tau) = 0$: in this way the *internal* 3-center of mass gets located in the centroid $x^\mu_s(\tau) = z^\mu(\tau, \vec{\sigma} = 0)$ of the Wigner hyperplane.

C. The external Poincare' group and the external center-of-mass variables on a Wigner hyper-plane.

The *external* viewpoint is proper to an arbitrary inertial Lorentz observer, describing the Wigner hyper-planes as leaves of a foliation of Minkowski space-time and determined by the time-like configurations of the isolated system. Therefore there is an *external* realization of the Poincaré group, whose Lorentz transformations rotate the Wigner hyper-planes and induce a Wigner rotation of the 3-vectors inside each Wigner hyperplane. As a consequence, each such hyperplane inherits an induced *internal Euclidean structure* and an *internal* unfaithful Euclidean action , with an associated unfaithful *internal* realization of the Poincare' group (the internal translations are generated by the first-class constraints $\vec{\kappa}_+ \approx 0$, so that they are eliminable gauge variables), which will be described in the next Subsection.

The *external* realization of the Poincaré generators with non-fixed invariants $\epsilon p^2_s = \epsilon^2_s \approx M^2_{sys}$ and $W^2 = -\epsilon p^2_s \vec{\bar{S}}^2_s \approx -\epsilon M^2_{sys} \vec{\bar{S}}^2$, is obtained from Eq.(3.8) (the four independent Hamiltonians [7] of this instant form, p^o_s and J^{oi}_s, are all functions *only* of the invariant mass M_{sys}, which contains the possible mutual interactions among the particles):

$$
\begin{aligned}
& p^\mu_s \quad , \qquad J^{\mu\nu}_s = \tilde{x}^\mu_s p^\nu_s - \tilde{x}^\nu_s p^\mu_s + \tilde{S}^{\mu\nu}_s, \\
& p^o_s = \sqrt{\epsilon^2_s + \vec{p}^2_s} = \epsilon_s \sqrt{1 + \vec{k}^2_s} \approx \sqrt{M^2_{sys} + \vec{p}^2_s} = M_{sys}\sqrt{1 + \vec{k}^2_s}, \qquad \vec{p}_s = \epsilon_s \vec{k}_s \approx M_{sys}\vec{k}_s,
\end{aligned}
$$

$$
\begin{aligned}
J_s^{ij} &= \tilde{x}_s^i p_s^j - \tilde{x}_s^j p_s^i + \delta^{ir}\delta^{js}\sum_{i=1}^N (\eta_i^r \kappa_i^s - \eta_i^s \kappa_i^r) = z_s^i k_s^j - z_s^j k_s^i + \delta^{ir}\delta^{js}\epsilon^{rsu}\bar{S}_s^u,\\
K_s^i &= J_s^{oi} = \tilde{x}_s^o p_s^i - \tilde{x}_s^i \sqrt{\epsilon_s^2 + \vec{p}_s^2} - \frac{1}{\epsilon_s + \sqrt{\epsilon_s^2 + \vec{p}_s^2}} \delta^{ir} p_s^s \sum_{i=1}^N (\eta_i^r \kappa_i^s - \eta_i^s \kappa_i^r) =\\
&= -\sqrt{1 + \vec{k}_s^2}\, z_s^i - \frac{\delta^{ir} k_s^s \epsilon^{rsu} \bar{S}_s^u}{1 + \sqrt{1 + \vec{k}_s^2}} \approx \tilde{x}_s^o p_s^i - \tilde{x}_s^i \sqrt{M_{sys}^2 + \vec{p}_s^2} - \frac{\delta^{ir} p_s^s \epsilon^{rsu} \bar{S}_s^u}{M_{sys} + \sqrt{M_{sys}^2 + \vec{p}_s^2}}.
\end{aligned}
\tag{3.13}
$$

Given a canonical realization of the ten Poincaré generators, one can build [18] three *external* 3-variables, the canonical 3-center of mass $\vec{q}_s$, the Moller 3-center of energy $\vec{R}_s$ and the Fokker-Pryce 3-center of inertia $\vec{Y}_s$ by using only these generators. For the rest-frame realization of the Poincaré algebra given in Eqs.(3.13) we get

$$
\begin{aligned}
\vec{R}_s &= -\frac{1}{p_s^o}\vec{K}_s = (\vec{\tilde{x}}_s - \frac{\vec{p}_s}{p_s^o}\tilde{x}_s^o) - \frac{\vec{\bar{S}}_s \times \vec{p}_s}{p_s^o(p_s^o + \epsilon_s)},\\
\vec{q}_s &= \vec{\tilde{x}}_s - \frac{\vec{p}_s}{p_s^o}\tilde{x}_s^o = \frac{\vec{z}_s}{\epsilon_s} = \vec{R}_s + \frac{\vec{\bar{S}}_s \times \vec{p}_s}{p_s^o(p_s^o + \epsilon_s)} = \frac{p_s^o \vec{R}_s + \epsilon_s \vec{Y}_s}{p_s^o + \epsilon_s},
\end{aligned}
$$

$$
\begin{aligned}
\vec{Y}_s &= \vec{q}_s + \frac{\vec{\bar{S}}_s \times \vec{p}_s}{\epsilon_s(p_s^o + \epsilon_s)} = \vec{R}_s + \frac{\vec{\bar{S}}_s \times \vec{p}_s}{p_s^o \epsilon_s},\\
&\{R_s^r, R_s^s\} = -\frac{1}{(p_s^o)^2}\epsilon^{rsu}\Omega_s^u, \qquad \vec{\Omega}_s = \vec{J}_s - \vec{R}_s \times \vec{p}_s,\\
&\{q_s^r, q_s^s\} = 0, \qquad \{Y_s^r, Y_s^s\} = \frac{1}{\epsilon_s p_s^o}\epsilon^{rsu}\Big[\bar{S}_s^u + \frac{\vec{\bar{S}}_s \cdot \vec{p}_s\, p_s^u}{\epsilon_s(p_s^o + \epsilon_s)}\Big],\\
\vec{p}_s \cdot \vec{q}_s &= \vec{p}_s \cdot \vec{R}_s = \vec{p}_s \cdot \vec{Y}_s = \vec{k}_s \cdot \vec{z}_s, \qquad \vec{p}_s = 0 \Rightarrow \vec{q}_s = \vec{Y}_s = \vec{R}_s.
\end{aligned}
\tag{3.14}
$$

All of these have the same velocity and coincide in the Lorentz rest frame where $\overset{\circ}{p}{}_s^\mu = \epsilon_s(1; \vec{0})$

Then, three *external* concepts of 4-center of mass can be defined (each having an *internal* 3-location inside the Wigner hyper-planes) starting from the kinematics of the Wigner hyper-planes and from the above concepts of 3-centers of mass [38] :

a) The *external* non-covariant canonical *4-center of mass* $\tilde{x}_s^\mu$ (with 3-location $\vec{\tilde{\sigma}}$), extension of the *canonical 3-position vector* $\vec{q}_s$ (also named *center of spin* [42]). $\vec{q}_s$ is the classical analogue of the Newton-Wigner position operator [39]. $\tilde{x}_s^\mu$ is a frame-dependent pseudovector ($\vec{q}_s$ does not satisfy the *world line condition* [38]), but it is canonical: $\{\tilde{x}_s^\mu, \tilde{x}_s^\nu\} = 0$.

b) The *external* non-covariant and non-canonical Møller *4-center of energy* R^μ_s (with 3-location $\vec{\sigma}_R$), extension of the Møller 3-center of mass $\vec{R}_s$ [40], which corresponds to the standard non-relativistic definition of center of mass of a system of particles with masses replaced by energies. $\vec{R}_s$ does not satisfy the world line condition, so that R^μ_s is a frame-dependent pseudovector and moreover $\{R^\mu_s, R^\nu_s\} \neq 0$.

c) The *external* non-canonical but covariant Fokker-Pryce *4-center of inertia* Y^μ_s (with 3-location $\vec{\sigma}_Y$), extension of the Fokker-Pryce 3-center of inertia [41, 42]. Y^μ_s is a 4-vector by construction: it is the Lorentz transform of the rest-frame pseudo-world-line $R_s^{(rest)\,\mu} = \tilde{x}_s^{(rest)\mu}$ of the Møller center of energy to an arbitrary frame. It holds $\{Y^\mu_s, Y^\nu_s\} \neq 0$.

Note that while the Fokker-Pryce Y^μ_s is the only 4-vector by construction, only $\tilde{x}^\mu_s(\tau)$ can be an *adapted coordinate in a Hamiltonian treatment with Dirac constraints*.

To find the 3-locations on the Wigner hyper-planes, with respect to the centroid world-line $x^\mu_s(\tau) = z^\mu(\tau, \vec{0})$, of these quantities, we note [11] that, since we have $T_s = u(p_s) \cdot x_s = u(p_s) \cdot \tilde{x}_s \equiv \tau$ on the Wigner hyperplane labeled by τ, we can require Y^μ_s and R^μ_s to have time components such that $u(p_s) \cdot Y_s = u(p_s) \cdot R_s = T_s \equiv \tau$. As a consequence, the following 3-locations are determined in Ref.[11]: a) for the world-line $Y^\mu_s(\tau) = z^\mu(\tau, \vec{\sigma}_Y) = \Big(\tilde{x}^o(\tau); \vec{Y}_s(\tau)\Big)$ we get $\sigma^r_Y = R^r_+$, with $\vec{R}_+$ defined in Eq.(3.16) of next Subsection; b) for the pseudo-world-line $\tilde{x}^\mu_s(\tau) = z^\mu(\tau, \tilde{\vec{\sigma}}) = \Big(\tilde{x}^o_s(\tau); \tilde{\vec{x}}_s(\tau)\Big)$ we get $\tilde{\sigma}^r = \sigma^r_Y + \frac{\bar{S}^{rv}_s\, u^v(p_s)}{1+u^o(p_s)}$; c) for the pseudo-world-line $R^\mu_s = z^\mu(\tau, \vec{\sigma}_R) = \Big(\tilde{x}^o(\tau); \vec{R}_s(\tau)\Big)$ we get $\sigma^r_R = \sigma^r_Y + \frac{[1-u^o(p_s)]\, \bar{S}^{rv}_s\, u^v(p_s)}{u^o(p_s)\,[1+u^o(p_s)]}$.

It is seen that *the external Fokker-Pryce non-canonical 4-center of inertia coincides with the centroid* $x^{(\vec{q}_+)\mu}_s(\tau)$ *carrying the internal 3-center of mass.*

In each Lorentz frame one has different pseudo-world-lines describing R^μ_s and $\tilde{x}^\mu_s$: the canonical 4-center of mass $\tilde{x}^\mu_s$ *lies in between* Y^μ_s and R^μ_s in every (non rest)-frame. In an arbitrary Lorentz frame, the pseudo-world-lines associated with $\tilde{x}^\mu_s$ and R^μ_s fill a world-tube (see the book in Ref.[40]) around the world-line Y^μ_s of the covariant non-canonical Fokker-Pryce 4-center of inertia Y^μ_s. The *invariant radius* of the tube is $\rho = \sqrt{-\epsilon W^2}/p^2 = |\vec{S}|/\sqrt{\epsilon p^2}$ where ($W^2 = -\epsilon p^2 \vec{S}^2$ is the Pauli-Lubanski invariant when $\epsilon p^2 > 0$). This classical intrinsic radius delimitates the non-covariance effects (the pseudo-world-lines) of the canonical 4-center of mass $\tilde{x}^\mu_s$. See Ref.[7] for a discussion of the properties of the *Møller radius*. At the quantum level ρ becomes the Compton wavelength of the isolated system times its spin eigenvalue $\sqrt{s(s+1)}$, $\rho \mapsto \hat{\rho} = \sqrt{s(s+1)}\hbar/M = \sqrt{s(s+1)}\lambda_M$ with $M = \sqrt{\epsilon p^2}$ the invariant mass and $\lambda_M = \hbar/M$ its Compton wavelength. The critique of classical relativistic physics argued from quantum pair production concerns testing of distances where, due to the Lorentz signature of space-time, intrinsic classical covariance problems emerge: the canonical 4-center of mass $\tilde{x}^\mu_s$ adapted to the first-class constraints of the system cannot be localized in a frame-independent way. Remember [6], finally, that ρ is also a remnant of the energy conditions of general relativity in flat Minkowski space-time: since the Møller non-canonical, non-covariant 4-center of energy R^μ has non-covariance properties localized inside the world-tube with radius ρ (see the book in [40]), it turns out that for an extended relativistic system with the material radius smaller of its intrinsic radius ρ one has: i) its peripheral rotation velocity can exceed the velocity of light; ii) its classical energy density cannot be positive definite everywhere in every frame.

D. The internal Poincare' group and the internal center-of-mass variables on a Wigner hyper-plane.

Let us consider now the notions defined according to the *internal* viewpoint. They correspond to an unfaithful *internal* realization of the Poincaré algebra: the *internal* 3-momentum $\vec{\kappa}_+$ vanishes due to the rest-frame conditions; the *internal* energy and angular momentum are given by the invariant mass M_{sys} and by the external spin (angular momentum with respect to $\tilde{x}^\mu_s(\tau)$) of the isolated system, respectively.

This *internal* realization of the Poincaré algebra is built inside the Wigner hyperplane by using the expression of $\bar{S}^{AB}_s$ given by Eq.(3.8) (the invariants are $M^2_{sys} - \vec{\kappa}^2_+ \approx M^2_{sys} > 0$ and $W^2 = -\epsilon(M^2_{sys} - \vec{\kappa}^2_+)\vec{\bar{S}}^2_s \approx -\epsilon M^2_{sys}\vec{\bar{S}}^2_s$; in the interacting case M_{sys} and $\vec{K}$ are modified by the mutual interactions among the particles)

$$
\begin{aligned}
&M_{sys} = H_M = \sum_{i=1}^N \sqrt{m_i^2 + \vec{\kappa}_i^2}, \qquad \vec{\kappa}_+ = \sum_{i=1}^N \vec{\kappa}_i \, (\approx 0),\\
&\vec{J} = \sum_{i=1}^N \vec{\eta}_i \times \vec{\kappa}_i, \qquad J^r = \bar{S}^r = \frac{1}{2}\epsilon^{ruv}\bar{S}^{uv} \equiv \bar{S}^r_s,\\
&\vec{K} = -\sum_{i=1}^N \sqrt{m_i^2 + \vec{\kappa}_i^2}\, \vec{\eta}_i, \qquad K^r = J^{or} = \bar{S}^{\tau r}_s.
\end{aligned}
\tag{3.15}
$$

As we shall see, $\vec{K} \approx 0$ are the natural gauge fixings for the first-class constraints $\vec{\kappa}_+ \approx 0$: this makes the internal realization even more unfaithful.

In analogy with the external viewpoint, the determination of the three *internal* 3-center of mass can be achieved using again the group theoretical methods of Ref.[18]:

i) a canonical *internal* 3-center of mass (or *3-center of spin*) $\vec{q}_+$;
ii) a non-canonical *internal* Møller *3-center of energy* $\vec{R}_+$;
iii) a non-canonical *internal* Fokker-Pryce *3-center of inertia* $\vec{y}_+$.

Starting from the *internal* realization (3.15) of the Poincaré algebra, we get the following Wigner spin 1 3-vectors: i) The *internal* Moller 3-center of energy $\vec{R}_+$ and the associated spin vector $\vec{S}_R$

$$
\begin{aligned}
\vec{R}_+ &= -\frac{1}{M_{sys}}\vec{K} = \frac{\sum_{i=1}^N \sqrt{m_i^2 + \vec{\kappa}_i^2}\, \vec{\eta}_i}{\sum_{k=1}^N \sqrt{m_k^2 + \vec{\kappa}_k^2}},\\
\vec{S}_R &= \vec{J} - \vec{R}_+ \times \vec{\kappa}_+,\\
&\{R^r_+, \kappa^s_+\} = \delta^{rs}, \qquad \{R^r_+, M_{sys}\} = \frac{\kappa^r_+}{M_{sys}}, \qquad \{R^r_+, R^s_+\} = -\frac{1}{M^2_{sys}}\epsilon^{rsu} S^u_R,\\
&\{S^r_R, S^s_R\} = \epsilon^{rsu}(S^u_R - \frac{1}{M^2_{sys}}\vec{S}_R \cdot \vec{\kappa}_+\, \kappa^u_+), \qquad \{S^r_R, M_{sys}\} = 0.
\end{aligned}
\tag{3.16}
$$

Note that the gauge fixing $\vec{R}_+ \approx 0$ gives

$$\vec{R}_+ \approx 0 \;\Rightarrow\; \dot{\vec{R}}_+ \circeq \{\vec{R}_+, H_D\} =$$
$$= \frac{\vec{\kappa}_+}{\sum_{k=1}^N \sqrt{m_k^2 + \vec{\kappa}_k^2}} - \vec{\lambda}(\tau) \frac{\sum_{i=1}^N \sqrt{m_i^2 + \vec{\kappa}_i^2}}{\sum_{k=1}^N \sqrt{m_k^2 + \vec{\kappa}_k^2}} \approx -\vec{\lambda}(\tau) \approx 0. \tag{3.17}$$

Furthermore, the *internal* boost generator of Eq.(3.15) may be rewritten as $\vec{K} = -M_{sys}\vec{R}_+$, so that $\vec{R}_+ \approx 0$ implies $\vec{K} \approx 0$.

ii) The canonical *internal* 3-center of mass $\vec{q}_+$ and the associated spin vector $\vec{S}_q$ $[\{q^r_+, q^s_+\} = 0$, $\{q^r_+, \kappa^s_+\} = \delta^{rs}$, $\{J^r, q^s_+\} = \epsilon^{rsu}\, q^u_+$, $\{S^r_q, S^s_q\} = \epsilon^{rsu} S^u_q]$

$$\begin{aligned}
\vec{q}_+ &= \vec{R}_+ - \frac{\vec{J} \times \vec{\Omega}}{\sqrt{M^2_{sys} - \vec{\kappa}^2_+}(M_{sys} + \sqrt{M^2_{sys} - \vec{\kappa}^2_+})} = \\
&= -\frac{\vec{K}}{\sqrt{M^2_{sys} - \vec{\kappa}^2_+}} + \frac{\vec{J} \times \vec{\kappa}_+}{\sqrt{M^2_{sys} - \vec{\kappa}^2_+}(M_{sys} + \sqrt{M^2_{sys} - \vec{\kappa}^2_+})} + \\
&+ \frac{\vec{K} \cdot \vec{\kappa}_+ \; \vec{\kappa}_+}{M_{sys}\sqrt{M^2_{sys} - \vec{\kappa}^2_+}\Big(M_{sys} + \sqrt{M^2_{sys} - \vec{\kappa}^2_+}\Big)}, \\
&\approx \vec{R}_+ \quad for \quad \vec{\kappa}_+ \approx 0; \qquad \{\vec{q}_+, M_{sys}\} = \frac{\vec{\kappa}_+}{M_{sys}},
\end{aligned}$$

$$\begin{aligned}
\vec{S}_q &= \vec{J} - \vec{q}_+ \times \vec{\kappa}_+ = \frac{M_{sys}\vec{J}}{\sqrt{M^2_{sys} - \vec{\kappa}^2_+}} + \\
&+ \frac{\vec{K} \times \vec{\kappa}_+}{\sqrt{M^2_{sys} - \vec{\kappa}^2_+}} - \frac{\vec{J} \cdot \vec{\kappa}_+ \; \vec{\kappa}_+}{\sqrt{M^2_{sys} - \vec{\kappa}^2_+}\Big(M_{sys} + \sqrt{M^2_{sys} - \vec{\kappa}^2_+}\Big)} \approx \vec{\bar{S}} = \vec{J}.
\end{aligned} \tag{3.18}$$

iii) The *internal* non-canonical Fokker-Pryce 3-center of inertia $\vec{y}_+$

$$\begin{aligned}
\vec{y}_+ &= \vec{q}_+ + \frac{\vec{S}_q \times \vec{\kappa}_+}{\sqrt{M^2_{sys} - \vec{\kappa}^2_+}(M_{sys} + \sqrt{M^2_{sys} - \vec{\kappa}^2_+})} = \vec{R}_+ + \frac{\vec{S}_q \times \vec{\kappa}_+}{M_{sys}\sqrt{M^2_{sys} - \vec{\kappa}^2_+}}, \\
\vec{q}_+ &= \vec{R}_+ + \frac{\vec{S}_q \times \vec{\kappa}_+}{M_{sys}(M_{sys} + \sqrt{M^2_{sys} - \vec{\kappa}^2_+})} = \frac{M_{sys}\vec{R}_+ + \sqrt{M^2_{sys} - \vec{\kappa}^2_+}\vec{y}_+}{M_{sys} + \sqrt{M^2_{sys} - \vec{\kappa}^2_+}}, \\
&\{y^r_+, y^s_+\} = \frac{1}{M_{sys}\sqrt{M^2_{sys} - \vec{\kappa}^2_+}} \epsilon^{rsu}\Big[S^u_q + \frac{\vec{S}_q \cdot \vec{\kappa}_+ \, \kappa^u_+}{\sqrt{M^2_{sys} - \vec{\kappa}^2_+}(M_{sys} + \sqrt{M^2_{sys} - \vec{\kappa}^2_+})}\Big].
\end{aligned} \tag{3.19}$$

Note that on the Wigner hyper-planes, due to the rest-frame conditions $\vec{\kappa}_+ \approx 0$, all the internal 3-centers of mass coincide, $\vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+$, and become essentially a unique *gauge* variable conjugate to $\vec{\kappa}_+$. As a natural gauge fixing for the rest-frame conditions $\vec{\kappa}_+ \approx 0$, we can add the vanishing of the *internal* Lorentz boosts $\vec{K} = -M_{sys}\,\vec{R}_+ \approx 0$: this implies $\vec{\lambda}(\tau) \approx 0$ and is equivalent to locate the internal canonical 3-center of mass $\vec{q}_+$ in $\vec{\sigma} = 0$, i.e. in the external centroid $x^\mu_s(\tau) = z^\mu(\tau, \vec{0})$ origin of the internal 3-coordinates in each Wigner hyperplane. Remember that the centroid $x^\mu_s(\tau)$ corresponds to the *unique* special-relativistic center-of-mass-like world-line of Refs.[46]. With these gauge fixings and with $T_s - \tau \approx 0$, the world-line $x^\mu_s(\tau)$ of the centroid becomes uniquely determined except for the arbitrariness in the choice of $x^\mu_s(0)$

$$x^{(\vec{q}_+)\mu}_s(\tau = T_s) = x^\mu_s(0) + u^\mu(p_s) T_s, \tag{3.20}$$

and coincides with the *external* covariant non-canonical Fokker-Pryce 4-center of inertia, $x^\mu_s(\tau) = x^\mu_s(0) + Y^\mu_s$. It can also be shown that the *centroid* $x^{(\mu)}_s(\tau)$ coincides with the *Dixon center of mass* of an extended object [29] as well as with the *Pirani* [47] and the *Tulczyjew* [48] *centroids*.

Note that in the non-relativistic limit all the quantities $\vec{q}_+$, $\vec{R}_+$, $\vec{y}_+$ tends the the non-relativistic center of mass $\vec{q}_{nr} = \frac{\sum_{i=1}^N m_i \vec{\eta}_i}{\sum_{i=1}^N m_i}$.

We are left with the problem of the construction of a canonical transformation bringing from the basis $\vec{\eta}_i$, $\vec{\kappa}_i$, to a new canonical basis $\vec{q}_+$, $\vec{\kappa}_+ (\approx 0)$, $\vec{\rho}_{q,a}$, $\vec{\pi}_{q,a}$, in which $\vec{S}_q = \sum_{a=1}^{N-1} \vec{\rho}_{q,a} \times \vec{\pi}_{q,a}$:

$$\begin{array}{|c|}\hline \vec{\eta}_i \\ \hline \vec{\kappa}_i \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|}\hline \vec{q}_+ & \vec{\rho}_{qa} \\ \hline \vec{\kappa}_+ & \vec{\pi}_{qa} \\ \hline \end{array} \tag{3.21}$$

Let us stress that this cannot be a point transformation, because of the momentum dependence of the relativistic internal 3-center of mass $\vec{q}_+$. The canonical transformation (3.21) will be constructed in the next Subsection by using the method of Gartenhaus-Schwartz [49] as delineated in Ref.[50].

In conclusion, at the relativistic level the non-relativistic Abelian translation symmetry generating the non-relativistic Noether constants $\vec{P} = const.$ gets split into the two following symmetries: i) the *external* Abelian translation symmetry whose Noether constants of motion are $\vec{p}_s = \epsilon_s \vec{k}_s \approx M_{sys} \vec{k}_s = const.$ (its conjugate variable being the *external* 3-center of mass $\vec{z}_s$); ii) the *internal* Abelian gauge symmetry generating the three first-class constraints $\vec{\kappa}_+ \approx 0$ (rest-frame conditions) inside the Wigner hyperplane (the conjugate *gauge* variable being the *internal* 3-center of mass $\vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+$). Of course, its non-relativistic counterpart is the non-relativistic rest-frame condition $\vec{P} \approx 0$.

E. The canonical transformation to the internal center-of-mass and relative variables for N free particles.

Since $\vec{q}_+$ and $\vec{\kappa}_+$ are known, we have only to find the internal conjugate variables appearing in the canonical transformation (3.21). They have been determined in Ref.[11] by using the technique of Ref.[49] and starting from a set of canonical variables defined in Ref.[6].

Precisely, starting from the naive *internal* center-of-mass variable $\vec{\eta}_+ = \frac{1}{N}\sum_{i=1}^N \vec{\eta}_i$, we applied with definition of relative variables $\vec{\rho}_a$, $\vec{\pi}_a$ based on the following family of point canonical transformations [the numerical parameters γ_{ai} satisfy the relations in Eqs.(2.2)]

$$\begin{array}{|c|}\hline \vec{\eta}_i \\ \hline \vec{\kappa}_i \\ \hline \end{array} \qquad \longrightarrow \qquad \begin{array}{|c|c|}\hline \vec{\eta}_+ & \vec{\rho}_a \\ \hline \vec{\kappa}_+ & \vec{\pi}_a \\ \hline \end{array} \qquad , \qquad a = 1, .., N-1,$$

$$\begin{aligned}
\vec{\eta}_i &= \vec{\eta}_+ + \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\rho}_a, \qquad \vec{\kappa}_i = \frac{1}{N} \vec{\kappa}_+ + \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\pi}_a,\\
\vec{\eta}_+ &= \frac{1}{N} \sum_{i=1}^N \vec{\eta}_i, \qquad \vec{\kappa}_+ = \sum_{i=1}^N \vec{\kappa}_i \approx 0,\\
\vec{\rho}_a &= \sqrt{N} \sum_{i=1}^N \gamma_{ai} \vec{\eta}_i, \qquad \vec{\pi}_a = \frac{1}{\sqrt{N}} \sum_{i=1}^N \gamma_{ai} \vec{\kappa}_i,\\
&\{\eta^r_i, \kappa^s_j\} = \delta_{ij} \delta^{rs}, \qquad \{\eta^r_+, \kappa^s_+\} = \delta^{rs}, \qquad \{\rho^r_a, \pi^s_b\} = \delta_{ab} \delta^{rs}.
\end{aligned} \tag{3.22}$$

Then, (in Appendix B of Ref.[12]), we gave the closed form of the canonical transformation for arbitrary N, which turned out to be *point in the momenta* but, unlike the non-relativistic case, *non-point* in the configurational variables.

Explicitly, for $N = 2$ we have

$$\begin{aligned}
M_{sys} &= \sqrt{m_1^2 + \vec{\kappa}_1^2} + \sqrt{m_1^2 + \vec{\kappa}_1^2}, \qquad \vec{S}_q = \vec{\rho}_q \times \vec{\pi}_q,\\
\vec{q}_+ &= \frac{\sqrt{m_1^2 + \vec{\kappa}_1^2}\, \vec{\eta}_1 + \sqrt{m_2^2 + \vec{\kappa}_2^2}\, \vec{\eta}_2}{\sqrt{M^2 - \vec{\kappa}_+^2}} + \frac{(\vec{\eta}_1 \times \vec{\kappa}_1 + \vec{\eta}_2 \times \vec{\kappa}_2) \times \vec{\kappa}_+}{\sqrt{M_{sys}^2 - \vec{\kappa}_+^2}\, (M_{sys} + \sqrt{M_{sys}^2 - \vec{\kappa}_+^2})} -\\
&- \frac{(\sqrt{m_1^2 + \vec{\kappa}_1^2}\, \vec{\eta}_1 + \sqrt{m_2^2 + \vec{\kappa}_2^2}\, \vec{\eta}_2) \cdot \vec{\kappa}_+\; \vec{\kappa}_+}{M_{sys} \sqrt{M_{sys}^2 - \vec{\kappa}_+^2}\, (M_{sys} + \sqrt{M_{sys}^2 - \vec{\kappa}_+^2})},\\
\vec{\kappa}_+ &= \vec{\kappa}_1 + \vec{\kappa}_2 \approx 0,\\
\vec{\pi}_q &= \vec{\pi} - \frac{\vec{\kappa}_+}{\sqrt{M_{sys}^2 - \vec{\kappa}_+^2}} \Big[\frac{1}{2} (\sqrt{m_1^2 + \vec{\kappa}_1^2} - \sqrt{m_2^2 + \vec{\kappa}_2^2}) -\\
&- \frac{\vec{\kappa}_+ \cdot \vec{\pi}}{\vec{\kappa}_+^2} (M_{sys} - \sqrt{M_{sys}^2 - \vec{\kappa}_+^2}) \Big] \approx \vec{\pi},\\
\vec{\rho}_q &= \vec{\rho} + \Big(\frac{\sqrt{m_1^2 + \vec{\kappa}_1^2}}{\sqrt{m_2^2 + \vec{\pi}_q^2}} + \frac{\sqrt{m_2^2 + \vec{\kappa}_2^2}}{\sqrt{m_1^2 + \vec{\pi}_q^2}} \Big) \frac{\vec{\kappa}_+ \cdot \vec{\rho}\; \vec{\pi}_q}{M_{sys} \sqrt{M_{sys}^2 - \vec{\kappa}_+^2}} \approx \vec{\rho},\\
\Rightarrow &\qquad M_{sys} = \sqrt{\mathcal{M}^2 + \vec{\kappa}_+^2} \approx \mathcal{M} = \sqrt{m_1^2 + \vec{\pi}_q^2} + \sqrt{m_2^2 + \vec{\pi}_q^2}.
\end{aligned} \tag{3.23}$$

The inverse canonical transformation is

$$
\begin{aligned}
\vec{\eta}_i &= \vec{q}_+ - \frac{\vec{S}_q \times \vec{\kappa}_+}{\sqrt{\mathcal{M}^2+\vec{\kappa}_+^2}\,(\mathcal{M}+\sqrt{\mathcal{M}^2+\vec{\kappa}_+^2}} + \frac{1}{2}\Big[(-)^{i+1} - \\
&- \frac{2\,\mathcal{M}\,\vec{\pi}_q\cdot\vec{\kappa}_+ + (m_1^2-m_2^2)\sqrt{\mathcal{M}^2+\vec{\kappa}_+^2}}{\mathcal{M}^2\sqrt{\mathcal{M}^2+\vec{\kappa}_+^2}}\Big].
\end{aligned}
$$

$$
\begin{aligned}
&\Big[\vec{\rho}_q - \frac{\vec{\rho}_q\cdot\vec{\kappa}_+\;\vec{\pi}_q}{\mathcal{M}\sqrt{\mathcal{M}^2+\vec{\kappa}_+^2}\Big(\frac{\sqrt{m_1^2+\vec{\kappa}_1^2}}{\sqrt{m_2^2+\vec{\pi}_q^2}}+\frac{\sqrt{m_2^2+\vec{\kappa}_2^2}}{\sqrt{m_1^2+\vec{\pi}_q^2}}\Big)^{-1}+\vec{\pi}_q\cdot\vec{\kappa}_+}\Big] \approx \\
&\approx\; \vec{q}_+ + \frac{1}{2}\Big[(-)^{i+1} - \frac{m_1^2-m_2^2}{\mathcal{M}^2}\Big]\vec{\rho}_q,\\
\vec{\kappa}_i \;&=\; \Big[\frac{1}{2} + \frac{(-)^{i+1}}{\mathcal{M}\sqrt{\mathcal{M}^2+\vec{\kappa}_+^2}}\Big(\vec{\pi}_q\cdot\vec{\kappa}_+\,[1-\frac{\mathcal{M}}{\vec{\kappa}_+^2}(\sqrt{\mathcal{M}^2+\vec{\kappa}_+^2}-\mathcal{M})] + \\
&+\; (m_1^2-m_2^2)\sqrt{\mathcal{M}^2+\vec{\kappa}_+^2}\Big)\Big]\,\vec{\kappa}_+ + (-)^{i+1}\,\vec{\pi}_q \approx (-)^{i+1}\,\vec{\pi}_q. \qquad (3.24)
\end{aligned}
$$

For $N \geq 2$, the Hamiltonian $M_{sys} = \sum_{i=1}^N \sqrt{m_i^2 + N\sum_{ab}^{1..N-1}\gamma_{ai}\,\gamma_{bi}\,\vec{\pi}_{qa}\cdot\vec{\pi}_{qb}}$ for the relative motions in the rest frame instant form, is a sum of square roots each containing a $(N-1)\times(N-1)$ matrix $K^{-1}_{(i)ab} = N\gamma_{ai}\gamma_{bi} = K^{-1}_{(i)ba}$ [note that in the non-relativistic limit only one such matrix appears, namely $k^{-1}_{ab} = \sum_{i=1}^N \frac{1}{m_i}K^{-1}_{(i)ab}$ of Eqs.(2.3)]. The existence of relativistic normal Jacobi coordinates would require the simultaneous diagonalization of these N matrices. This is impossible, however, because

$$
[K^{-1}_{(i)}, K^{-1}_{(j)}]_{ab} = G_{(ij)ab} = -G_{(ij)ba} = -G_{(ji)ab} = -N[\gamma_{ai}\gamma_{bj} - \gamma_{aj}\gamma_{bi}]. \qquad (3.25)
$$

There are $\frac{1}{2}N(N-1)$ matrices G_{ij}, each one with $\frac{1}{2}(N-1)(N-2)$ independent elements. While the conditions $G_{(ij)ab}=0$ are $\frac{1}{4}N(N-1)^2(N-2)$, the free parameters at our disposal in the γ_{ai} are only $\frac{1}{2}(N-1)(N-2)$. For N=3, there are 3 conditions and only 1 parameter; for N=4, 18 conditions and 3 parameters.

In conclusion: it is impossible to diagonalize the N quadratic forms under the square roots simultaneously, *no relativistic normal Jacobi coordinates exist, and reduced masses and inertia tensor cannot be defined.*

The relative Lagrangian can be worked out in the special case of N=2 with equal masses ($m_1 = m_2 = m$). It results

$$
L_{rel}(\vec{\rho}, \dot{\vec{\rho}}) = -m\sqrt{4 - \dot{\vec{\rho}}^{\,2}}. \qquad (3.26)
$$

so that the relative velocity is bounded by $|\dot{\vec{\rho}}| \leq 2$.

Let us write $\vec{\rho} = \rho\hat{\rho}$ with $\rho = |\vec{\rho}|$ and $\hat{\rho} = \frac{\vec{\rho}}{|\vec{\rho}|}$. With a single relative variable the three Euler angles θ^α are redundant: there are only two independent angles, identifying the position of the unit 3-vector $\hat{\rho}$ on S^2. We shall use the parametrization (Euler angles $\theta^1 = \phi$, $\theta^2 = \theta$, $\theta^3 = 0$)

$$\hat{\rho}^r = R^{rs}(\theta,\phi)\,\hat{\rho}^s_o = \Big(R_z(\theta)R_y(\phi)\Big)^{rs}\,\hat{\rho}^s_o, \qquad \hat{\rho}_o = (0,0,1). \tag{3.27}$$

In analogy with Subsection IA of the non-relativistic case, we get the following body frame velocity and angular velocity (ρ is the only shape variable for N=2)

$$\begin{aligned}
\check{v}^r &= R^{T\,rs}\dot{\rho}^s = \rho(R^T\dot{R})^{rs}\hat{\rho}^s_o + \dot{\rho}\hat{\rho}^r_o = \rho\epsilon^{ru3}\check{\omega}^u + \dot{\rho}\hat{\rho}^r_o = \rho(\vec{\omega}\times\hat{\rho}_o)^r + \dot{\rho}\hat{\rho}^r_o,\\
\vec{\omega} &= (\frac{1}{2}\,\epsilon^{urs}\,(R^T\dot{R})^{rs}) = (\check{\omega}^1 = -sin\,\theta\dot{\phi}, \check{\omega}^2 = \dot{\theta}, 0),\\
\vec{v}^2 &= \check{I}(\rho)\vec{\omega}^2 + \dot{\rho}^2, \qquad \check{I}(\rho) = \rho^2.
\end{aligned} \tag{3.28}$$

The non-relativistic inertia tensor of the dipole $\check{I}_{nr} = \mu\rho^2$ ($\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass) is replaced by $\check{I} = \check{I}_{nr}/\mu = \rho^2$. The Lagrangian in anholonomic variables become

$$\tilde{L}(\vec{\omega},\rho,\dot{\rho}) = -m\sqrt{4 - \check{I}(\rho)\vec{\omega}^2 - \dot{\rho}^2}. \tag{3.29}$$

It is clear that the bound $|\dot{\vec{\rho}}| \leq 2$ puts upper limitations upon the kinetic energy of both the rotational and vibrational motions.

The canonical momenta are

$$\vec{S} = \frac{\partial\tilde{L}}{\partial\vec{\omega}} = \frac{m\check{I}(\rho)\vec{\omega}}{\sqrt{4 - \check{I}(\rho)\vec{\omega}^2 - \dot{\rho}^2}}, \qquad \pi = \frac{\partial\tilde{L}}{\partial\dot{\rho}} = \frac{m\dot{\rho}}{\sqrt{4 - \check{I}(\rho)\vec{\omega}^2 - \dot{\rho}^2}}. \tag{3.30}$$

Note that *there is no more a linear relation between spin and angular velocity.*

When $|\dot{\vec{\rho}}|$ varies between 0 and 2 the momenta vary between 0 and ∞, so that *in phase space there is no bound from the limiting light velocity.* This shows once more that in special relativity it is convenient to work in the Hamiltonian framework avoiding relative and angular velocities.

Since we have $\sqrt{4 - \check{I}(\rho)\vec{\omega}^2 - \dot{\rho}^2} = \frac{2m}{\sqrt{m^2 + \check{I}^{-1}(\rho)\vec{S}^2 + \pi^2}}$, the inversion formulas become

$$\begin{aligned}
\vec{\omega} &= \frac{\vec{S}}{m\check{I}(\rho)}\sqrt{4 - \check{I}(\rho)\vec{\omega}^2 - \dot{\rho}^2} = \frac{2\check{I}^{-1}(\rho)\vec{S}}{\sqrt{m^2 + \check{I}^{-1}(\rho)\vec{S}^2 + \pi^2}},\\
\dot{\rho} &= \frac{\pi}{m}\sqrt{4 - \check{I}(\rho)\vec{\omega}^2 - \dot{\rho}^2} = \frac{2\pi}{\sqrt{m^2 + \check{I}^{-1}(\rho)\vec{S}^2 + \pi^2}}.
\end{aligned} \tag{3.31}$$

Then, the Hamiltonian results

$$M_{sys} = \pi\dot{\rho} + \vec{S}\cdot\vec{\omega} - \tilde{L} = 2\sqrt{m^2 + \check{I}^{-1}(\rho)\vec{S}^2 + \pi^2}. \tag{3.32}$$

F. Dynamical body frames and canonical spin bases for N relativistic particles.

For isolated systems the constraint manifold [7] is a stratified manifold with each stratum corresponding to a type of Poincaré orbit. The main stratum (dense in the constraint manifold) corresponds to all configurations of the isolated system belonging to time-like Poincaré orbits with $\epsilon p_s^2 \approx \epsilon M_{sys}^2 > 0$. As said in Ref.[20], this implies that the internal 3-center-of-mass coordinates have been adapted to the co-adjoint orbits of the internal realization Poincaré group. But, since the second Poincaré invariant (the Pauli-Lubanski invariant $\vec{W}_s^2 = -p_s^2 \vec{S}_s^2$) does not appear among the canonical variables, this canonical basis is not adapted, as yet, to a *typical form* of canonical action of the Poincaré group [18] on the phase space of the isolated system. However, remember that the *scheme A* for the *internal* realization of the Poincaré group [18] contains the canonical pairs $\vec{\kappa}_+$, $\vec{q}_+$, S_q^3, $arctg\, \frac{S_q^2}{S_q^1}$, and the two Casimirs invariants $|\vec{S}_q| = \sqrt{-W^2/M_{sys}^2}$, $M_{sys} = \sum_{i=1}^N \sqrt{m_i^2 + \vec{\kappa}_i^2}$.

As a consequence, as shown in Ref.[20], it is possible to construct a canonical basis including both Poincaré invariants in such a way that all of the internal coordinates are adapted to the co-adjoint action of the group and the new internal relative variables are thereby adapted to the SO(3) group.

In conclusion, in the rest-frame instant form of dynamics *the construction of the internal dynamical body frames and of the internal canonical spin bases is identical to that of the non-relativistic case.* Only the form of the relative Hamiltonian, the invariant mass M_{sys}, is modified.

G. Action-at-a-distance interacting particles and relativistic orbit theory.

As shown in Ref.[6] and its bibliography (see also Ref.[51]), the action-at-a-distance interactions inside the Wigner hyperplane may be introduced under either the square roots (scalar and vector potentials) or outside (scalar potential like the Coulomb one) appearing in the free Hamiltonian. Since a Lagrangian density in presence of action-at-a-distance mutual interactions is not known and since we are working in an instant form of dynamics, the potentials in the constraints restricted to hyper-planes must be introduced *by hand*. The only restriction is that the Poisson brackets of the modified constraints must generate the same algebra of the free ones.

In the rest-frame instant form the most general Hamiltonian with action-at-a-distance interactions is

$$M_{sys,int} = \sum_{i=1}^{N} \sqrt{m_i^2 + U_i + [\vec{k}_i - \vec{V}_i]^2} + V, \tag{3.33}$$

where $U = U(\vec{\kappa}_k, \vec{\eta}_h - \vec{\eta}_k)$, $\vec{V}_i = \vec{V}_i(\vec{k}_{j\neq i}, \vec{\eta}_i - \vec{\eta}_{j\neq i})$, $V = V_o(|\vec{\eta}_i - \vec{\eta}_j|) + V'(\vec{k}_i, \vec{\eta}_i - \vec{\eta}_j)$.

If we use the canonical transformation (3.21) defining the relativistic canonical internal 3-center of mass (now it is interaction-dependent, $\vec{q}_+^{(int)}$) and relative variables on the Wigner hyperplane, with the rest-frame conditions $\vec{\kappa}_+ \approx 0$, the rest frame Hamiltonian for the relative motion becomes

$$M_{sys,int} \approx \sum_{i=1}^{N} \sqrt{m_i^2 + \tilde{U}_i + [\sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\pi}_{qa} - \tilde{\vec{V}}_i]^2} + \tilde{V}, \tag{3.34}$$

where

$$\begin{aligned}
\tilde{U}_i &= U([\sqrt{N} \sum_{a=1}^{N-1} \gamma_{ak} \vec{\pi}_{qa}, \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ah} - \gamma_{ak}) \vec{\rho}_{qa}),\\
\tilde{\vec{V}}_i &= \vec{V}_i([\sqrt{N} \sum_{a=1}^{N-1} \gamma_{aj\neq i} \vec{\pi}_{qa}, \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ai} - \gamma_{aj\neq i}) \vec{\rho}_{qa}),\\
\tilde{V} &= V_o(|\frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ai} - \gamma_{aj}) \vec{\rho}_{qa}|) + V'([\sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\pi}_{qa}, \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ai} - \gamma_{aj}) \vec{\rho}_{qa}).
\end{aligned} \tag{3.35}$$

In order to build a realization of the internal Poincare' group, besides $M_{sys,int}$ we need to know the potentials appearing in the boosts $\vec{K}_{int}$ (being an instant form, $\vec{\kappa}_+ \approx 0$ and $\vec{J}$ are the free ones). We need therefore the rest-frame energy-momentum tensor of the isolated system (see later).

Since the 3-centers $\vec{R}_+$ and $\vec{q}_+$ become interaction dependent, the final canonical basis $\vec{q}_+$, $\vec{\kappa}_+$, $\vec{\rho}_{qa}$, $\vec{\pi}_{qa}$ is not explicitly known the interacting case. For an isolated system, however, we have $M_{sys} = \sqrt{\mathcal{M}^2 + \vec{\kappa}_+^2} \approx \mathcal{M}$ with $\mathcal{M}$ independent of $\vec{q}_+$ ($\{M_{sys}, \vec{\kappa}_+\} = 0$ in the internal Poincare' algebra). This suggests that the same result should hold true even in the interacting case. Indeed, by its definition, the Gartenhaus-Schwartz transformation gives $\vec{\rho}_{qa} \approx \vec{\rho}_a$, $\vec{\pi}_{qa} \approx \vec{\pi}_a$ also in presence of interactions, so that we get

$$\begin{aligned}
M_{sys,int}|_{\vec{\kappa}_+=0} &= \Big(\sum_i \sqrt{m_i^2 + U_i + (\vec{\kappa}_i - \vec{V}_i)^2} + V\Big)|_{\vec{\kappa}_+=0} = \sqrt{\mathcal{M}_{int}^2 + \vec{\kappa}_+^2}|_{\vec{\kappa}_+} =\\
&= \mathcal{M}_{int}|_{\vec{\kappa}_+=0} = \sum_i \sqrt{m_i^2 + \tilde{U}_i + (\vec{\kappa}_i - \tilde{\vec{V}}_i)^2} + \tilde{V},
\end{aligned} \tag{3.36}$$

where the potentials $\tilde{U}_i$, $\tilde{\vec{V}}_i$, $\tilde{V}$ are now functions of $\vec{\pi}_{qa} \cdot \vec{\pi}_{qb}$, $\vec{\pi}_{qa} \cdot \vec{\rho}_{qb}$, $\vec{\rho}_{qa} \cdot \vec{\rho}_{qb}$.

Unlike in the non-relativistic case, the canonical transformation (3.23) is now *interaction dependent* (and no more point in the momenta), since $\vec{q}_+$ is determined by a set of Poincare' generators depending on the interactions. The only thing to do in the generic situation is therefore to use the free relative variables (3.23) even in the interacting case. We cannot impose anymore, however, the natural gauge fixings $\vec{q}_+ \approx 0$ ($\vec{K} \approx 0$) of the free case, since it is replaced by $\vec{q}_+^{(int)} \approx 0$ (namely by $\vec{K}_{int} \approx 0$), the only gauge fixing identifying the centroid with the external Fokker-Pryce 4-center of inertia also in the interacting case. Once written in terms of the canonical variables (3.23) of the free case, these equations can be solved for $\vec{q}_+$, which takes a form $\vec{q}_+ \approx \vec{f}(\vec{\rho}_{aq}, \vec{\pi}_{aq})$ as a consequence of the potentials appearing in the boosts. Therefore, for $N = 2$, the reconstruction of the relativistic orbit by means of

Eqs.(3.24) in terms of the relative motion is given by (similar equations hold for arbitrary N)

$$\vec{\eta}_i(\tau) \approx \vec{q}_+(\vec{\rho}_q, \vec{\pi}_q) + \frac{1}{2}\Big[(-)^{i+1} - \frac{m_1^2 - m_2^2}{\mathcal{M}^2}\Big]\, \vec{\rho}_q \rightarrow_{c\rightarrow\infty} \frac{1}{2}\Big[(-)^{i+1} - \frac{m_1 - m_2}{m}\Big]\, \vec{\rho}_q,$$

$$\Rightarrow \qquad x_i^\mu(\tau) = z^\mu_{wigner}(\tau, \vec{\eta}_i(\tau)) = x^\mu_s(0) + u^\mu(p_s)\,\tau + \epsilon^\mu_r(u(p_s))\,\eta^r_i(\tau). \tag{3.37}$$

While the potentials in $M_{sys,int}$ determine $\vec{\rho}_q(\tau)$ and $\vec{\pi}_q(\tau)$ through Hamilton equations, the potentials in $\vec{K}_{int}$ determine $\vec{q}_+(\vec{\rho}_q, \vec{\pi}_q)$. It is seen, therefore - as it should be expected - that the relativistic theory of orbits is much more complicated than in the non-relativistic case, where the absolute orbits $\vec{\eta}_i(t)$ are proportional to the relative orbit $\vec{\rho}_q(t)$.

A relevant example of this type of isolated system has been studied in the second paper of Ref.[6] starting from the isolated system of N charged positive-energy particles (with Grassmann-valued electric charges $Q_i = \theta^*\,\theta$, $Q_i^2 = 0$, $Q_i\,Q_j = Q_j\,Q_i \neq 0$ for $i \neq j$) plus the electro-magnetic field. After a Shanmugadhasan canonical transformation, this system can be expressed only in terms of transverse Dirac observables corresponding to a radiation gauge for the electro-magnetic field. The expression of the energy-momentum tensor in this gauge will be shown in the next Subsection IVC (where $\vec{K}_{int}$ can be calculated). In the semi-classical approximation of the second paper of Ref.[6], the electro-magnetic degrees of freedom are re-expressed in terms of the particle variables by means of the Lienard-Wiechert solution, in the framework of the rest-frame instant form. In this way the exact semi-classical relativistic form of the action-at-a-distance Darwin potential in the reduced phase space of the particles has been obtained. Note that this form is independent of the choice of the Green function in the Lienard-Wiechert solution. In the second paper of Ref.[6] the associated energy-momentum tensor for the case $N = 2$ [Eqs.(6.48)] is also given. The internal energy is $M_{sys} = \sqrt{\mathcal{M}^2 + \vec{\kappa}_+^2} \approx \mathcal{M} = \sum_{i=1}^2 \sqrt{m_i^2 + \vec{\pi}^2} + \frac{Q_1\,Q_2}{4\pi\,\rho}\,[1 + \tilde{V}(\vec{\pi}^2, \vec{\pi}\cdot\frac{\vec{\rho}}{\rho})]$ where $\tilde{V}$ is given in Eqs.(6.34), (6.35) [in Eqs. (6.36), (6.37) for $m_1 = m_2$]. The internal boost $\vec{K}_{int}$ [Eq.(6.46)] allows the determination of the 3-center of energy $\vec{R}_+ = -\frac{\vec{K}_{int}}{M_{sys}} \approx \vec{q}_+ \approx \vec{y}_+$ in the present interacting case.

H. An Example of Deformable Continuous System: the Classical Klein-Gordon Field.

Consider now the problem of separating the relativistic center of mass for isolated special-relativistic *extended systems*. In Ref.[46], mainly devoted to the same problem in general relativity, it is shown that in special relativity there is a unique world-line describing this notion, corresponding to the centroid $x^\mu_s(\tau)$ of the rest-frame instant form. The first attempt to define a *collective variable* for a relativistic extended body in the Hamiltonian formulation was done in Ref.[25] in the case of a configuration of the Klein-Gordon field (which is used, e.g., in the description of bosonic stars). Let us show how the previous kinematical formalism is working in this case.

The rest-frame instant form for a classical real free Klein-Gordon field [26] $\phi(\tau, \vec{\sigma}) = \tilde{\phi}(z^\mu(\tau, \vec{\sigma}))$ ($\tilde{\phi}(x)$ is the standard field) on the Wigner hyper-planes is built starting from the reformulation of Klein-Gordon action as a parametrized Minkowski theory. The relevant quantities, as well as the internal and external Poincare' generators, can be worked out as in Section III, Subsection A, B, C and D, for N free relativistic particles [$q^A = (q^\tau = \omega(q);\, q^r)$, $\omega = \sqrt{m^2 + \vec{q}^2}$, $\Omega(q) = (2\pi)^3\, 2\,\omega(q)$, $d\tilde{q} = d^3q/\Omega(q)$]

$$
\begin{aligned}
a\left(\tau,\vec{q}\right) &= \int d^3\sigma\, \left[\omega\left(q\right)\phi\left(\tau,\vec{\sigma}\right) + i\pi\left(\tau,\vec{\sigma}\right)\right] e^{i(\omega(q)\tau - \vec{q}\cdot\vec{\sigma})},\\
a^*\left(\tau,\vec{q}\right) &= \int d^3\sigma\, \left[\omega\left(q\right)\phi\left(\tau,\vec{\sigma}\right) - i\pi\left(\tau,\vec{\sigma}\right)\right] e^{-i(\omega(q)\tau - \vec{q}\cdot\vec{\sigma})},\\
\phi\left(\tau,\vec{\sigma}\right) &= \int d\tilde{q}\, \left[a\left(\tau,\vec{q}\right) e^{-i(\omega(q)\tau - \vec{q}\cdot\vec{\sigma})} + a^*\left(\tau,\vec{q}\right) e^{+i(\omega(q)\tau - \vec{q}\cdot\vec{\sigma})}\right],\\
\pi\left(\tau,\vec{\sigma}\right) &= -i\int d\tilde{q}\,\omega\left(q\right) \left[a\left(\tau,\vec{q}\right) e^{-i(\omega(q)\tau - \vec{q}\cdot\vec{\sigma})} - a^*\left(\tau,\vec{q}\right) e^{+i(\omega(\vec{q})\tau - \vec{q}\cdot\vec{\sigma})}\right],
\end{aligned}
$$

$$
\{\phi(\tau\vec{\sigma}), \pi(\tau,\vec{\sigma}_1\} = \delta^3(\vec{\sigma} - \vec{\sigma}_1), \qquad \left\{a\left(\tau,\vec{q}\right), a^*\left(\tau,\vec{k}\right)\right\} = -i\Omega\left(q\right)\delta^3\left(\vec{q} - \vec{k}\right). \tag{3.38}
$$

$$
\begin{aligned}
M_\phi &= P^\tau_\phi = \frac{1}{2}\int d^3\sigma \Big[\pi^2 + (\vec{\partial}\phi)^2 + m^2\phi^2\Big](\tau,\vec{\sigma}) = \int d\tilde{q}\,\omega\left(q\right) a^*\left(\tau,\vec{q}\right) a\left(\tau,\vec{q}\right),\\
\vec{P}_\phi &= \int d^3\sigma[\pi\vec{\partial}\phi](\tau,\vec{\sigma}) = \int d\tilde{q}\,\vec{q}\, a^*\left(\tau,\vec{q}\right) a\left(\tau,\vec{q}\right) \approx 0,
\end{aligned}
$$

$$
\begin{aligned}
J^{rs}_\phi &= \int d^3\sigma[\pi(\sigma^r\partial^s - \sigma^s\partial^r)\phi](\tau,\vec{\sigma}) =\\
&= -i\int d\tilde{q}\, a^*\left(\tau,\vec{q}\right)\left(q^r\frac{\partial}{\partial q^s} - q^s\frac{\partial}{\partial q^r}\right) a\left(\tau,\vec{q}\right),\\
J^{\tau r}_\phi &= -\tau P^r_\phi + \frac{1}{2}\int d^3\sigma\,\sigma^r\, [\pi^2 + (\vec{\partial}\phi)^2 + m^2\phi^2](\tau,\vec{\sigma}) =\\
&= -\tau P^r_\phi + i\int d\tilde{q}\,\omega\left(q\right) a^*\left(\tau,\vec{q}\right)\frac{\partial}{\partial q^r} a\left(\tau,\vec{q}\right),
\end{aligned}
$$

$$
\begin{aligned}
&p^\mu_s\\
J^{ij}_s &= \tilde{x}^i_s p^j_s - \tilde{x}^j_s p^i_s + \delta^{ir}\delta^{js}\bar{S}^{rs}_s, \qquad J^{oi}_s = \tilde{x}^o_s p^i_s - \tilde{x}^i_s p^o_s - \frac{\delta^{ir}\bar{S}^{rs}_s p^s_s}{p^o_s + \epsilon_s},\\
\bar{S}^{rs}_s &\equiv S^{rs}_\phi = J^{rs}_\phi|_{\vec{P}_\phi = 0} = \int d^3\sigma\{\sigma^r[\pi\partial^s\phi](\tau,\vec{\sigma}) - (r \leftrightarrow s)\}|_{\vec{P}_\phi = 0}.
\end{aligned} \tag{3.39}
$$

Here $\tilde{x}^\mu_s$ is the usual canonical non-covariant 4-center of mass of Eq.(3.9). We are working on Wigner hyper-planes with $\tau \equiv T_s$, so that the Hamiltonian (3.12) is $H_D = M_\phi - \vec{\lambda}(\tau)\cdot\vec{P}_\phi$.

We want to construct four variables $X^A_\phi[\phi,\pi] = (X^\tau_\phi; \vec{X}_\phi)$ canonically conjugated to $P^A_\phi[\phi,\pi] = (P^\tau_\phi; \vec{P}_\phi)$. First of all we make a canonical transformation to *modulus-phase* canonical variables

$$
\begin{aligned}
a\left(\tau,\vec{q}\right) &= \sqrt{I\left(\tau,\vec{q}\right)}\, e^{[i\varphi(\tau,\vec{q})]}, \qquad I\left(\tau,\vec{q}\right) = a^*\left(\tau,\vec{q}\right) a\left(\tau,\vec{q}\right),\\
a^*\left(\tau,\vec{q}\right) &= \sqrt{I\left(\tau,\vec{q}\right)}\, e^{[-i\varphi(\tau,\vec{q})]}, \qquad \varphi\left(\tau,\vec{q}\right) = \frac{1}{2i}\ln\left[\frac{a\left(\tau,\vec{q}\right)}{a^*\left(\tau,\vec{q}\right)}\right],\\
&\{I\left(\tau,\vec{q}\right), \varphi\left(\tau,\vec{q}'\right)\} = \Omega\left(q\right)\delta^3\left(\vec{q}-\vec{q}'\right).
\end{aligned}
\tag{3.40}
$$

In terms of the original canonical variables ϕ, π, we have

$$
\begin{aligned}
I(\tau,\vec{q}) &= \int d^3\sigma \int d^3\sigma'\, e^{i\vec{q}\cdot(\vec{\sigma}-\vec{\sigma}')}[\omega(q)\phi(\tau,\vec{\sigma}) - i\pi(\tau,\vec{\sigma})][\omega(q)\phi(\tau,\vec{\sigma}') + i\pi(\tau,\vec{\sigma}')],\\
\varphi(\tau,\vec{q}) &= \frac{1}{2i}\ln\Big[\frac{\int d^3\sigma\,[\omega(q)\phi(\tau,\vec{\sigma}) + i\pi(\tau,\vec{\sigma})]e^{i(\omega(q)\tau-\vec{q}\cdot\vec{\sigma})}}{\int d^3\sigma'\,[\omega(q)\phi(\tau,\vec{\sigma}') - i\pi(\tau,\vec{\sigma}')]e^{-i(\omega(q)\tau-\vec{q}\cdot\vec{\sigma}')}}\Big] =\\
&= \omega(q)\,\tau + \frac{1}{2i}\ln\Big[\frac{\int d^3\sigma\,[\omega(q)\phi(\tau,\vec{\sigma}) + i\pi(\tau,\vec{\sigma})]e^{-i\vec{q}\cdot\vec{\sigma}}}{\int d^3\sigma'\,[\omega(q)\phi(\tau,\vec{\sigma}') - i\pi(\tau,\vec{\sigma}')]e^{i\vec{q}\cdot\vec{\sigma}'}}\Big],
\end{aligned}
$$

$$
\begin{aligned}
\phi\left(\tau,\vec{\sigma}\right) &= \int d\tilde{q}\,\sqrt{I\left(\tau,\vec{q}\right)}\,\Big[e^{i\varphi(\tau,\vec{q})-i(\omega(q)\tau-\vec{q}\cdot\vec{\sigma})} + e^{-i\varphi(\tau,\vec{q})+i(\omega(q)\tau-\vec{q}\cdot\vec{\sigma})}\Big],\\
\pi\left(\tau,\vec{\sigma}\right) &= -i\int d\tilde{q}\,\omega\left(q\right)\sqrt{I\left(\tau,\vec{q}\right)}\,\Big[e^{i\varphi(\tau,\vec{q})-i(\omega(q)\tau-\vec{q}\cdot\vec{\sigma})} - e^{-i\varphi(\tau,\vec{q})+i(\omega(q)\tau-\vec{q}\cdot\vec{\sigma})}\Big].
\end{aligned}
\tag{3.41}
$$

Then, the internal Poincaré charges of the field configuration take the form

$$
\begin{aligned}
P^\tau_\phi &= \int d\tilde{q}\,\omega\left(q\right) I\left(\tau,\vec{q}\right), \qquad \vec{P}_\phi = \int d\tilde{q}\,\vec{q}\, I\left(\tau,\vec{q}\right) \approx 0,\\
J^{rs}_\phi &= \int d\tilde{q}\, I\left(\tau,\vec{q}\right)\left(q^r\frac{\partial}{\partial q^s} - q^s\frac{\partial}{\partial q^r}\right)\varphi\left(\tau,\vec{q}\right),\\
J^{\tau r}_\phi &= -\tau P^r_\phi - \int d\tilde{q}\,\omega\left(q\right) I\left(\tau,\vec{q}\right)\frac{\partial}{\partial q^r}\varphi\left(\tau,\vec{q}\right),
\end{aligned}
\tag{3.42}
$$

while the classical analogue of the occupation number is $[\triangle = -\vec{\partial}^2]$

$$
\begin{aligned}
N_\phi &= \int d\tilde{q}\, a^*(\tau,\vec{q})a(\tau,\vec{q}) = \int d\tilde{q}\, I(\tau,\vec{q}) =\\
&= \frac{1}{2}\int d^3\sigma[\pi\frac{1}{\sqrt{m^2+\triangle}} + \phi\sqrt{m^2+\triangle}]\phi(\tau,\vec{\sigma}).
\end{aligned}
\tag{3.43}
$$

1. Definition of the collective variables.

Define the four functionals of the phases

$$
\begin{aligned}
X^\tau_\phi &= \int d\tilde{q}\omega(q) F^\tau(q)\,\varphi(\tau,\vec{q}), \qquad \vec{X}_\phi = \int d\tilde{q}\vec{q}\,F(q)\,\varphi(\tau,\vec{q}),\\
&\Rightarrow \{X^r_\phi, X^s_\phi\} = 0, \qquad \{X^\tau_\phi, X^r_\phi\} = 0.
\end{aligned} \tag{3.44}
$$

depending on two Lorentz scalar functions $F^\tau(q)$, $F(q)$. Their form will be restricted by the following requirements implying that X^A_ϕ and P^A_ϕ are canonical variables

$$
\left\{P^\tau_\phi, X^\tau_\phi\right\} = 1,\ \left\{P^r_\phi, X^s_\phi\right\} = -\delta^{rs},\ \left\{P^r_\phi, X^\tau_\phi\right\} = 0,\ \left\{P^\tau_\phi, X^r_\phi\right\} = 0, \tag{3.45}
$$

Since $\left\{P^\tau_\phi, X^\tau_\phi\right\} = \int d\tilde{q}\omega^2(q)\,F^\tau(q)$ and $\left\{P^r_\phi, X^s_\phi\right\} = \int d\tilde{q}\,q^r q^s\,F(q)$, we must impose the following normalizations for $F^\tau(q)$, $F(q)$

$$
\int d\tilde{q}\omega^2(q) F^\tau(q) = 1, \qquad \int d\tilde{q}\,q^r q^s F(q) = -\delta^{rs}. \tag{3.46}
$$

Moreover, $\left\{P^r_\phi, X^\tau_\phi\right\} = \int d\tilde{q}\omega(q) q^r F^\tau(q)$ and $\left\{P^\tau_\phi, X^r_\phi\right\} = \int d\tilde{q}\omega(q)\,q^r F(q)$, imply the conditions

$$
\int d\tilde{q}\omega(q)\,q^r F^\tau(q) = 0, \qquad \int d\tilde{q}\omega(q)\,q^r F(q) = 0. \tag{3.47}
$$

that are automatically satisfied since $F^\tau(q)$, $F(q)$, $q = |\vec{q}|$, are even under $q^r \rightarrow -q^r$.

A solution of Eqs.(3.47) is

$$
F^\tau(q) = \frac{16\pi^2}{m\,q^2\sqrt{m^2+q^2}} e^{-\frac{4\pi}{m^2}q^2}, \qquad F(q) = -\frac{48\pi^2}{m\,q^4}\sqrt{m^2+q^2}\,e^{-\frac{4\pi}{m^2}q^2}. \tag{3.48}
$$

The singularity in $\vec{q} = 0$ requires $\varphi(\tau,\vec{q}) \rightarrow_{q\rightarrow 0} q^\eta$, $\eta > 0$ for the existence of X^τ_ϕ, $\vec{X}_\phi$.

Note that for field configurations $\phi(\tau,\vec{\sigma})$ such that the Fourier transform $\hat{\phi}(\tau,\vec{q})$ has compact support in a sphere centered at $\vec{q} = 0$ of volume V, we get $X^\tau_\phi = -\frac{1}{V}\int \frac{d^3q}{\omega(q)}\varphi(\tau,\vec{q})$, $\vec{X}_\phi = \frac{1}{V}\int d^3q \frac{3\vec{q}}{\vec{q}^2}\varphi(\tau,\vec{q})$.

2. Auxiliary relative variables.

As in Ref.[25], let us define an auxiliary relative action variable and an auxiliary relative phase variable

$$
\begin{aligned}
\hat{I}(\tau,\vec{q}) &= I(\tau,\vec{q}) - F^\tau(q)\,P^\tau_\phi\omega(q) + F(q)\,\vec{q}\cdot\vec{P}_\phi,\\
\hat{\varphi}(\tau,\vec{q}) &= \varphi(\tau,\vec{q}) - \omega(q)\,X^\tau_\phi + \vec{q}\cdot\vec{X}_\phi.
\end{aligned} \tag{3.49}
$$

The previous canonicity conditions on $F^\tau(q)$, $F(q)$, imply

$$\int d\tilde{q}\, \omega(q)\, \hat{I}(\tau, \vec{q}) = 0, \qquad \int d\tilde{q}\, q^i\, \hat{I}(\tau, \vec{q}) = 0,$$
$$\int d\tilde{q} F^\tau(\mathrm{q})\, \omega(q)\, \hat{\varphi}(\tau, \vec{q})\, d\tilde{q} = 0, \qquad \int d\tilde{q} F(q)\, q^i\, \hat{\varphi}(\tau, \vec{q}) = 0. \tag{3.50}$$

Such auxiliary variables have the following non-zero Poisson brackets

$$\begin{aligned}\left\{\hat{I}\left(\tau, \vec{k}\right), \hat{\varphi}(\tau, \vec{q})\right\} &= \Delta\left(\vec{k}, \vec{q}\right) = \\ &= \Omega(k)\, \delta^3\left(\vec{k} - \vec{q}\right) - F^\tau(k)\, \omega(k)\, \omega(q) + F(k)\, \vec{k} \cdot \vec{q}.\end{aligned} \tag{3.51}$$

The distribution $\Delta\left(\vec{k}, \vec{q}\right)$ has the semigroup property and satisfies the four constraints

$$\int d\tilde{q} \Delta\left(\vec{k}, \vec{q}\right) \Delta\left(\vec{q}, \vec{k}'\right) = \Delta\left(\vec{k}, \vec{k}'\right),$$
$$\int d\tilde{q}\, \omega(q)\, \Delta\left(\vec{q}, \vec{k}\right) = 0, \qquad \int d\tilde{q}\, q^r\, \Delta\left(\vec{q}, \vec{k}\right) = 0,$$
$$\int d\tilde{q}\, F^\tau(q)\, \omega(q)\, \Delta\left(\vec{k}, \vec{q}\right) = 0, \qquad \int d\tilde{q}\, q^r\, F(q)\, \Delta\left(\vec{k}, \vec{q}\right) = 0. \tag{3.52}$$

At this stage the canonical variables $I(\tau, \vec{q})$, $\varphi(\tau, \vec{q})$ for the Klein-Gordon field are replaced by the non-canonical set X^τ_ϕ , P^τ_ϕ , $\vec{X}_\phi$, $\vec{P}_\phi$, $\hat{I}(\tau, \vec{q})$, $\hat{\varphi}(\tau, \vec{q})$ with Poisson brackets

$$\begin{aligned}
&\left\{P^\tau_\phi, X^\tau_\phi\right\} = 1, \quad \left\{P^r_\phi, X^s_\phi\right\} = -\delta^{rs}, \quad \left\{P^r_\phi, X^\tau_\phi\right\} = 0, \quad \left\{P^\tau_\phi, X^r_\phi\right\} = 0,\\
&\left\{X^r_\phi, X^s_\phi\right\} = 0, \quad \left\{X^\tau_\phi, X^r_\phi\right\} = 0, \quad \left\{P^A_\phi, P^B_\phi\right\} = 0, \quad A, B = (\tau, r),\\
&\left\{P^\tau_\phi, \hat{I}(\tau, \vec{q})\right\} = 0, \quad \left\{P^r_\phi, \hat{I}(\tau, \vec{q})\right\} = 0, \quad \left\{\hat{I}(\tau, \vec{q}), X^\tau_\phi\right\} = 0, \quad \left\{\hat{I}(\tau, \vec{q}), X^r_\phi\right\} = 0,\\
&\left\{X^r_\phi, \hat{\varphi}(\tau, \vec{q})\right\} = 0, \quad \left\{X^\tau_\phi, \hat{\varphi}(\tau, \vec{q})\right\} = 0, \quad \left\{P^r_\phi, \hat{\varphi}(\tau, \vec{q})\right\} = 0, \quad \left\{P^\tau_\phi, \hat{\varphi}(\tau, \vec{q})\right\} = 0,\\
&\left\{\hat{I}\left(\tau, \vec{k}\right), \hat{\varphi}(\tau, \vec{q})\right\} = \Omega(k)\, \delta^3\left(\vec{k} - \vec{q}\right) - F^\tau(k)\, \omega(k)\, \omega(q) + F(k)\, \vec{k} \cdot \vec{q}.
\end{aligned} \tag{3.53}$$

Note finally that the generators of the internal Lorentz group are already decomposed into the collective and the relative parts, each satisfying the Lorentz algebra and having vanishing mutual Poisson brackets

$$\begin{aligned}
J^{rs}_\phi &= L^{rs}_\phi + \hat{S}^{rs}_\phi,\\
&\qquad L^{rs}_\phi = X^r_\phi P^s_\phi - X^s_\phi P^r_\phi, \qquad \hat{S}^{rs}_\phi = \int d\tilde{q}\, \hat{I}(\tau, \vec{q}) \left(q^r \frac{\partial}{\partial q^s} - q^s \frac{\partial}{\partial q^r}\right) \hat{\varphi}(\tau, \vec{q}),\\
J^{\tau r}_\phi &= L^{\tau r}_\phi + \hat{S}^{\tau r}_\phi,\\
&\qquad L^{\tau r}_\phi = [X^\tau_\phi - \tau] P^r_\phi - X^r_\phi P^\tau_\phi, \qquad \hat{S}^{\tau r}_\phi = -\int d\tilde{q}\, \omega(q)\, \hat{I}(\tau, \vec{q}) \frac{\partial}{\partial q^r} \hat{\varphi}(\tau, \vec{q}).
\end{aligned} \tag{3.54}$$

3. Canonical relative variables.

Now we must find the relative canonical variables hidden inside the auxiliary ones. They are not free but satisfy Eqs.(3.50). As in Ref.[25], let us introduce the following differential operator [$\triangle_{LB}$ is the Laplace-Beltrami operator of the mass shell sub-manifold H_3^1 (see Ref.[25, 26])]

$$\begin{aligned}\mathcal{D}_{\vec{q}} &= 3 - m^2 \triangle_{LB} = \\ &= 3 - m^2 \left[\sum_{i=1}^{3} \left(\frac{\partial}{\partial q^i} \right)^2 + \frac{2}{m^2} \sum_{i=1}^{3} q^i \frac{\partial}{\partial q^i} + \frac{1}{m^2} \left(\sum_{i=1}^{3} q^i \frac{\partial}{\partial q^i} \right)^2 \right].\end{aligned} \tag{3.55}$$

Note that, being invariant under Wigner's rotations, is a scalar on the Wigner hyperplane.

Since ω (q) and $\vec{q}$ are null modes of this operator [25], we can put

$$\hat{I}\left(\tau, \vec{q}\right) = \mathcal{D}_{\vec{q}} \mathbf{H}(\tau, \vec{q}), \qquad \mathbf{H}\left(\tau, \vec{q}\right) = \int d\tilde{k}\, \mathcal{G}\left(\vec{q}, \vec{k}\right) \hat{I}\left(\tau, \vec{k}\right), \tag{3.56}$$

with $\mathcal{G}\left(\vec{q}, \vec{k}\right)$ the Green function of $\mathcal{D}_{\vec{q}}$ (see Refs.[25] for its expression)

$$\mathcal{D}_{\vec{q}} \mathcal{G}\left(\vec{q}, \vec{k}\right) = \Omega\left(k\right) \delta^3\left(\vec{k} - \vec{q}\right). \tag{3.57}$$

Like in Ref.[25], for each zero mode $f_o(\vec{q})$ of $\mathcal{D}_{\vec{q}}$ [$\mathcal{D}_{\vec{q}}\, f_o(\vec{q}) = 0$] for which $| \int d\tilde{q}\, f_o(\vec{q}) \hat{I}(\tau, \vec{q})| < \infty$, integrating by parts, we get

$$\begin{aligned}\int d\tilde{q} \quad & f_o(\vec{q}) \hat{I}(\tau, \vec{q}) = \int d\tilde{q}\, f_o(\vec{q}) \mathcal{D}_{\vec{q}} \mathbf{H}(\tau, \vec{q}) = \\ &= -\frac{1}{2(2\pi)^3} \int d^3q\, \frac{\partial}{\partial q^r} \Big(\frac{m^2 \delta^{rs} + q^r q^s}{\omega(q)} \Big[f_o(\vec{q}) \frac{\partial}{\partial q^s} \mathbf{H}(\tau, \vec{q}) - \mathbf{H}(\tau, \vec{q}) \frac{\partial}{\partial q^s} f_o(\vec{q}) \Big] \Big).\end{aligned} \tag{3.58}$$

The boundary conditions (ensuring finite Poincaré generators)

$$\begin{aligned}\mathbf{H}(\tau, \vec{q}) &\rightarrow_{q \rightarrow 0} q^{-1+\epsilon}, \qquad \epsilon > 0, \\ \mathbf{H}(\tau, \vec{q}) &\rightarrow_{q \rightarrow \infty} q^{-3-\sigma}, \qquad \sigma > 0,\end{aligned} \tag{3.59}$$

imply $\int d\tilde{q}\, f_o(\vec{q}) \hat{I}(\tau, \vec{q}) = 0$ (so that the first two conditions (3.50) are also satisfied), or

$$\int d\tilde{q}\, f_o(\vec{q}) I(\tau, \vec{q}) = P^\tau_\phi \int d\tilde{q}\, \omega(q) f_o(\vec{q}) F^\tau(q) - \vec{P}_\phi \cdot \int d\tilde{q}\, \vec{q}\, f_o(\vec{q}) F(q). \tag{3.60}$$

It is shown in Ref.[25] that, by restricting ourselves to field configurations for which $I(\tau, \vec{q}) \rightarrow_{q \rightarrow 0} q^{-3+\eta}$ with $\eta \in (0, 1]$ and by imposing the following restriction on $\phi(\tau, \vec{\sigma})$ and $\pi(\tau, \vec{\sigma}) = \partial_\tau \phi(\tau, \vec{\sigma})$

$$
\begin{aligned}
P_{l0} \;&=\; const. \int d\tilde{q}\, q^l\, {}_2F_1(\frac{l-1}{2}, \frac{l+3}{2}; l+\frac{3}{2}; -q^2)\, Y_{l0}(\theta,\varphi) \\
&\int d^3\sigma \int d^3\sigma' e^{i\vec{q}\cdot(\vec{\sigma}-\vec{\sigma}')} \Big[(m^2+q^2)\phi(\tau,\vec{\sigma})\phi(\tau,\vec{\sigma}') + \pi(\tau,\vec{\sigma})\pi(\tau,\vec{\sigma}')\Big] = 0. \qquad (3.61)
\end{aligned}
$$

we can identify the class of the Klein-Gordon field configurations that is compatible with the previous canonical transformation and lead to a unique realization of the Poincaré group without any ambiguity

We can satisfy the constraints (3.50) on $\hat{\varphi}(\tau,\vec{q})$ with the definition $[\mathcal{D}_{\vec{q}}\omega(q) = \mathcal{D}_{\vec{q}}\vec{q} = 0]$

$$
\begin{aligned}
\hat{\varphi}(\tau,\vec{q}) \;&=\; \int d\tilde{k} \int d\tilde{k}' \mathbf{K}(\tau,\vec{k}) \mathcal{G}(\vec{k},\vec{k}') \Delta(\vec{k}',\vec{q}), \\
\mathbf{K}(\tau,\vec{q}) \;&=\; \mathcal{D}_{\vec{q}}\hat{\varphi}(\tau,\vec{q}) = \mathcal{D}_{\vec{q}}\varphi(\tau,\vec{q}), \\
&\quad \rightarrow_{q\rightarrow\infty} q^{1-\epsilon}, \quad \epsilon > 0, \qquad \rightarrow_{q\rightarrow 0} q^{\eta-2}, \quad \eta > 0, \qquad (3.62)
\end{aligned}
$$

which also imply

$$
\begin{aligned}
&\left\{\mathbf{H}(\tau,\vec{q}), X_\phi^\tau\right\} = 0, \qquad \left\{\mathbf{H}(\tau,\vec{q}), P_\phi^\tau\right\} = 0, \\
&\left\{\mathbf{H}(\tau,\vec{q}), X_\phi^r\right\} = 0, \qquad \left\{\mathbf{H}(\tau,\vec{q}), P_\phi^r\right\} = 0, \\
&\left\{\mathbf{K}(\tau,\vec{q}), X_\phi^\tau\right\} = 0, \qquad \left\{\mathbf{K}(\tau,\vec{q}), P_\phi^\tau\right\} = 0, \\
&\left\{\mathbf{K}(\tau,\vec{q}), X_\phi^r\right\} = 0, \qquad \left\{\mathbf{K}(\tau,\vec{q}), P_\phi^r\right\} = 0, \\
&\left\{\mathbf{H}(\tau,\vec{q}), \mathbf{K}(\tau,\vec{q}')\right\} = \Omega(q)\,\delta^3(\vec{q}-\vec{q}'). \qquad (3.63)
\end{aligned}
$$

The final decomposition of the internal Lorentz generators is

$$
\begin{aligned}
J_\phi^{rs} \;&=\; L_\phi^{rs} + S_\phi^{rs}, \\
&\quad L_\phi^{rs} = X_\phi^r P_\phi^s - X_\phi^s P_\phi^r, \qquad S_\phi^{rs} = \int d\tilde{k} \mathbf{H}\left(\tau,\vec{k}\right) \left(k^r \frac{\partial}{\partial k^s} - k^s \frac{\partial}{\partial k^r}\right) \mathbf{K}\left(\tau,\vec{k}\right), \\
J_\phi^{\tau r} \;&=\; L_\phi^{\tau r} + S_\phi^{\tau r}, \\
&\quad L_\phi^{\tau r} = (X_\phi^\tau - \tau) P_\phi^r - X_\phi^r P_\phi^\tau, \qquad S_\phi^{\tau r} = -\int d\tilde{q} \omega(q)\, \mathbf{H}(\tau,\vec{q}) \frac{\partial}{\partial q^r} \mathbf{K}(\tau,\vec{q}). \qquad (3.64)
\end{aligned}
$$

4. Field variables in terms of collective-relative variables.

We have found the canonical transformation

$$
\begin{aligned}
I(\tau,\vec{q}) \;&=\; F^\tau(q)\omega(q) P_\phi^\tau - F(q)\vec{q}\cdot\vec{P}_\phi + \mathcal{D}_{\vec{q}}\mathbf{H}(\tau,\vec{q}), \\
\varphi(\tau,\vec{q}) \;&=\; \int d\tilde{k} \int d\tilde{k}' \mathbf{K}(\tau,\vec{k}) \mathcal{G}\left(\vec{k},\vec{k}'\right) \Delta\left(\vec{k}',\vec{q}\right) + \omega(q)\, X_\phi^\tau - \vec{q}\cdot\vec{X}_\phi,
\end{aligned}
$$

$$
\begin{aligned}
N_\phi &= P^\tau_\phi \int d\tilde{q}\, \omega(q) F^\tau(q) - \vec{P}_\phi \cdot \int d\tilde{q}\, \vec{q} F(q) + \int d\tilde{q}\, \mathcal{D}_{\vec{q}} \mathbf{H}(\tau, \vec{q}) = \\
&= \tilde{c} \frac{P^\tau_\phi}{m} + \int d\tilde{q}\, \mathcal{D}_{\vec{q}} \mathbf{H}(\tau, \vec{q}), \\
&\quad \tilde{c} = \tilde{c}(m) = m \int d\tilde{q} \omega(q) F^\tau(q) = 2 \int_0^\infty \frac{dq}{\sqrt{m^2+q^2}} e^{-\frac{4\pi}{m^2} q^2} = 2 e^{4\pi} \int_m^\infty \frac{dx}{\sqrt{x^2-m^2}} e^{-\frac{4\pi}{m^2} x^2},
\end{aligned}
\tag{3.65}
$$

with the two functions $F^\tau(q)$, $F(q)$ given in Eqs.(3.48). Its inverse is

$$
\begin{aligned}
P^\tau_\phi &= \int d\tilde{q} \omega(q) I(\tau, \vec{q}) = \frac{1}{2} \int d^3\sigma \Big[\pi^2 + (\vec{\partial}\phi)^2 + m^2 \phi^2 \Big](\tau, \vec{\sigma}), \\
\vec{P}_\phi &= \int d\tilde{q} \vec{q} I(\tau, \vec{q}) = \int d^3\sigma \Big[\pi \vec{\partial} \phi \Big](\tau, \vec{\sigma}), \\
X^\tau_\phi &= \int d\tilde{q} \omega(q) F^\tau(q) \varphi(\tau, \vec{q}) = \tau + \\
&+ \frac{1}{2\pi i m} \int d^3q \frac{e^{-\frac{4\pi}{m^2} q^2}}{q^2\, \omega(q)} ln \Big[\frac{\omega(q) \int d^3\sigma e^{i\vec{q}\cdot\vec{\sigma}} \phi(\tau, \vec{\sigma}) + i \int d^3\sigma e^{i\vec{q}\cdot\vec{\sigma}} \pi(\tau, \vec{\sigma})}{\omega(q) \int d^3\sigma e^{-i\vec{q}\cdot\vec{\sigma}} \phi(\tau, \vec{\sigma}) - i \int d^3\sigma e^{-i\vec{q}\cdot\vec{\sigma}} \pi(\tau, \vec{\sigma})} \Big] = \\
&\overset{def}{=} \tau + \tilde{X}^\tau_\phi, \qquad \Rightarrow \quad L^{\tau r}_\phi = \tilde{X}^\tau_\phi P^r_\phi - X^r_\phi P^\tau_\phi, \\
\vec{X}_\phi &= \int d\tilde{q} \vec{q} F(q) \varphi(\tau, \vec{q}) = \\
&= \frac{2i}{\pi m} \int d^3q\, \frac{q^i}{q^4}\, e^{-\frac{4\pi}{m^2} q^2} ln \Big[\frac{\sqrt{m^2+q^2} \int d^3\sigma e^{i\vec{q}\cdot\vec{\sigma}} \phi(\tau, \vec{\sigma}) + i \int d^3\sigma e^{i\vec{q}\cdot\vec{\sigma}} \pi(\tau, \vec{\sigma})}{\sqrt{m^2+q^2} \int d^3\sigma e^{-i\vec{q}\cdot\vec{\sigma}} \phi(\tau, \vec{\sigma}) - i \int d^3\sigma e^{-i\vec{q}\cdot\vec{\sigma}} \pi(\tau, \vec{\sigma})} \Big],
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{H}(\tau, \vec{q}) &= \int d\tilde{k} \mathcal{G}(\vec{q}, \vec{k}) [I(\tau, \vec{k}) - F^\tau(k) \omega(k) \int d\tilde{q}_1 \omega(q_1) I(\tau, \vec{q}_1) + \\
&+ F(k) \vec{k} \cdot \int d\tilde{q}_1 \vec{q}_1\, I(\tau, \vec{q}_1)] = \\
&= \int d^3\sigma_1 d^3\sigma_2 \Big[\pi(\tau, \vec{\sigma}_1) \pi(\tau, \vec{\sigma}_2) \int d\tilde{k} \mathcal{G}(\vec{q}, \vec{k}) \int d\tilde{k}_1 \triangle(\vec{k}, \vec{k}_1) e^{i\vec{k}_1\cdot(\vec{\sigma}_1 - \vec{\sigma}_2)} + \\
&+ \phi(\tau, \vec{\sigma}_1) \phi(\tau, \vec{\sigma}_2) \int d\tilde{k} \mathcal{G}(\vec{q}, \vec{k}) \int d\tilde{k}_1 \omega^2(k_1) \triangle(\vec{k}, \vec{k}_1) e^{i\vec{k}_1\cdot(\vec{\sigma}_1 - \vec{\sigma}_2)} - \\
&- i \Big(\pi(\tau, \vec{\sigma}_1) \phi(\tau, \vec{\sigma}_2) + \pi(\tau, \vec{\sigma}_2) \phi(\tau, \vec{\sigma}_1) \Big) \\
&\quad \int d\tilde{k} \mathcal{G}(\vec{q}, \vec{k}) \int d\tilde{k}_1 \omega(k_1) \triangle(\vec{k}.\vec{k}_1) e^{i\vec{k}_1\cdot(\vec{\sigma}_1 - \vec{\sigma}_2)} \Big], \\
\mathbf{K}(\tau, \vec{q}) &= \mathcal{D}_{\vec{q}} \hat{\varphi}(\tau, \vec{q}) = \mathcal{D}_{\vec{q}} \varphi(\tau, \vec{q}) = \\
&= \frac{1}{2i} \mathcal{D}_{\vec{q}} \ln \left[\frac{\int d^3\sigma\ [\omega\,(q)\,\phi\,(\tau, \vec{\sigma}) + i\pi\,(\tau, \vec{\sigma})]\, e^{-i\vec{q}\cdot\vec{\sigma}}}{\int d^3\sigma'\ [\omega\,(q)\,\phi\,(\tau, \vec{\sigma}') - i\pi\,(\tau, \vec{\sigma}')]\, e^{i\vec{q}\cdot\vec{\sigma}'}} \right].
\end{aligned}
\tag{3.66}
$$

The remaining canonical variables $a(\tau,\vec{q})$, $\phi(\tau,\vec{\sigma})$, $\pi(\tau,\vec{\sigma})$, can be worked out in terms of the final ones

$$
\begin{aligned}
a(\tau,\vec{q}) &= \sqrt{F^\tau(q)\omega(q)P^\tau_\phi - F(q)\vec{q}\cdot\vec{P}_\phi + \mathcal{D}_{\vec{q}}\mathbf{H}(\tau,\vec{q})} \\
&\quad e^{i[\omega(q)X^\tau_\phi - \vec{q}\cdot\vec{X}_\phi] + i\int d\tilde{k}\int d\tilde{k}'\,\mathbf{K}(\tau,\vec{k})\mathcal{G}\big(\vec{k},\vec{k}'\big)\Delta\big(\vec{k}',\vec{q}\big)}, \\
N_\phi &= \tilde{c}\frac{P^\tau_\phi}{m} - + \int d\tilde{k}\,\mathcal{D}_{\vec{k}}\mathbf{H}(\tau,\vec{k}),
\end{aligned}
\tag{3.67}
$$

$$
\begin{aligned}
\phi(\tau,\vec{\sigma}) &= \int d\tilde{q}\sqrt{F^\tau(q)\omega(q)P^\tau_\phi - F(q)\vec{q}\cdot\vec{P}_\phi + \mathcal{D}_{\vec{q}}\mathbf{H}(\tau,\vec{q})} \\
&\quad \Big[e^{-i[\omega(q)\big(\tau - X^\tau_\phi\big) - \vec{q}\cdot\big(\vec{\sigma}-\vec{X}_\phi\big)] + i\int d\tilde{k}\int d\tilde{k}'\,\mathbf{K}(\tau,\vec{k})\mathcal{G}\big(\vec{k},\vec{k}'\big)\Delta\big(\vec{k}',\vec{q}\big)} + \\
&\quad + e^{i[\omega(q)\big(\tau - X^\tau_\phi\big) - \vec{q}\cdot\big(\vec{\sigma}-\vec{X}_\phi\big)] - i\int d\tilde{k}\int d\tilde{k}'\,\mathbf{K}\big(\tau,\vec{k}\big)\mathcal{G}\big(\vec{k},\vec{k}'\big)\Delta\big(\vec{k}',\vec{q}\big)}\Big] = \\
&= 2\int d\tilde{q}\mathbf{A}_{\vec{q}}(\tau;P^A_\phi,\mathbf{H}]\, cos\, \Big[\vec{q}\cdot\vec{\sigma} + \mathbf{B}_{\vec{q}}(\tau;X^A_\phi,\mathbf{K}]\Big],
\end{aligned}
$$

$$
\begin{aligned}
\pi(\tau,\vec{\sigma}) &= -i\int d\tilde{q}\omega(q)\sqrt{F^\tau(q)\omega(q)P^\tau_\phi - F(q)\vec{q}\cdot\vec{P}_\phi + \mathcal{D}_{\vec{q}}\mathbf{H}(\tau,\vec{q})} \\
&\quad \Big[e^{-i[\omega(q)\big(\tau - X^\tau_\phi\big) - \vec{q}\cdot\big(\vec{\sigma}-\vec{X}_\phi\big)] + i\int d\tilde{k}\int d\tilde{k}'\,\mathbf{K}(\tau,\vec{k})\mathcal{G}\big(\vec{k},\vec{k}'\big)\Delta\big(\vec{k}',\vec{q}\big)} - \\
&\quad - e^{+i[\omega(q)\big(\tau - X^\tau_\phi\big) - \vec{q}\cdot\big(\vec{\sigma}-\vec{X}_\phi\big)] - i\int d\tilde{k}\int d\tilde{k}'\,\mathbf{K}\big(\tau,\vec{k}\big)\mathcal{G}\big(\vec{k},\vec{k}'\big)\Delta\big(\vec{k}',\vec{q}\big)}\Big] = \\
&= -2\int d\tilde{q}\omega(q)\mathbf{A}_{\vec{q}}(\tau;P^A_\phi,\mathbf{H}]\, sin\, \Big[\vec{q}\cdot\vec{\sigma} + \mathbf{B}_{\vec{q}}(\tau;X^A_\phi,\mathbf{K}]\Big],
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}_{\vec{q}}(\tau;P^A_\phi,\mathbf{H}] &= \sqrt{F^\tau(q)\omega(q)P^\tau_\phi - F(q)\vec{q}\cdot\vec{P}_\phi + \mathcal{D}_{\vec{q}}\mathbf{H}(\tau,\vec{q})} = \sqrt{I(\tau,\vec{q})}, \\
\mathbf{B}_{\vec{q}}(\tau;X^A_\phi,\mathbf{K}] &= -\vec{q}\cdot\vec{X}_\phi - \omega(q)(\tau - X^\tau_\phi) + \int d\tilde{k}d\tilde{k}'\mathbf{K}(\tau,\vec{k})\mathcal{G}(\vec{k},\vec{k}')\triangle(\vec{k}',\vec{q}) = \\
&= \varphi(\tau,\vec{q}) - \omega(q)\tau.
\end{aligned}
\tag{3.68}
$$

Summarizing, the Klein-Gordon field configuration is described by:

i) its energy $P^\tau_\phi = M_\phi$, i.e. its invariant mass, and the conjugate field time-variable $X^\tau_\phi \stackrel{def}{=} \tau + \tilde{X}^\tau_\phi$ ($\tau \equiv T_s$), which is equal to τ plus some kind of *internal time* $\tilde{X}^\tau_\phi$ (note that in the N-body case this variable does not exist: M_{sys} is a given function of the other canonical variables and not an independent canonical variable like here);
ii) the conjugate reduced canonical variables of a free point $\vec{X}_\phi$, $\vec{P}_\phi \approx 0$ (see $\vec{\eta}_+$ and $\vec{\kappa}_+ \approx 0$ of Eqs.(3.22) in the N-body case);
iii) an infinite set of canonically conjugate relative variables $\mathbf{H}(\tau,\vec{q})$, $\mathbf{K}(\tau,\vec{q})$ ($\vec{\rho}_a$ and $\vec{\pi}_a$ of Eq.(3.22) in the N-body case).

While the set iii) describes an infinite set of *canonical relative variables* with respect to the relativistic collective variables of the sets i) and ii), the sets i) and ii) describe a *monopole* field configuration, which depends only on 8 degrees of freedom like a scalar particle at rest $[\vec{P}_\phi \approx 0]$ with mass $M_s = \sqrt{(P^\tau_\phi)^2 - \vec{P}^2_\phi} \approx P^\tau_\phi$ but without the mass-shell condition $M_s \approx const.$, corresponding to the decoupled collective variables of the field configuration. The conditions $\mathbf{H}(\tau, \vec{q}) = \mathbf{K}(\tau, \vec{q}) = 0$ select the class of field configurations, solutions of the Klein-Gordon equation, which are of the *monopole* type on the Wigner hyperplanes

$$
\begin{aligned}
\phi_{mon}(\tau, \vec{\sigma}) &= 2 \int d\tilde{q} \sqrt{F^\tau(q)\omega(q)P^\tau_\phi - F(q)\vec{q}\cdot\vec{P}_\phi} cos \Big[\vec{q}\cdot(\vec{\sigma} - \vec{X}_\phi) - \omega(q)(\tau - X^\tau_\phi)\Big] \approx \\
&\approx 2\sqrt{P^\tau_\phi} \int d\tilde{q} \sqrt{F^\tau(q)\omega(q)} cos \Big[\vec{q}\cdot(\vec{\sigma} - \vec{X}_\phi) - \omega(q)(\tau - X^\tau_\phi)\Big], \\
\pi_{mon}(\tau, \vec{\sigma}) &= -2 \int d\tilde{q} \sqrt{F^\tau(q)\omega(q)P^\tau_\phi - F(q)\vec{q}\cdot\vec{P}_\phi} sin \Big[\vec{q}\cdot(\vec{\sigma} - \vec{X}_\phi) - \omega(q)(\tau - X^\tau_\phi)\Big] \approx \\
&\approx -2\sqrt{P^\tau_\phi} \int d\tilde{q} \sqrt{F^\tau(q)\omega(q)} sin \Big[\vec{q}\cdot(\vec{\sigma} - \vec{X}_\phi) - \omega(q)(\tau - X^\tau_\phi)\Big]. \qquad (3.69)
\end{aligned}
$$

If we add the gauge-fixings $\vec{X}_\phi \approx 0$ to $\vec{P}_\phi \approx 0$ (this implies $\vec{\lambda}(\tau) = 0$ and $H_D = M_\phi$) and go to Dirac brackets, the rest-frame instant-form canonical variables of the Klein-Gordon field in the gauge $\tau \equiv T_s$ are (in the following formulas it holds $T_s - X^\tau_\phi = -\tilde{X}^\tau_\phi$)

$$
\begin{aligned}
a(T_s, \vec{q}) &= \sqrt{F^\tau(q)\omega(q)P^\tau_\phi + \mathcal{D}_{\vec{q}}\mathbf{H}(T_s, \vec{q})} \\
&\quad e^{i[\omega(q)\tilde{X}^\tau_\phi + \vec{q}\cdot\vec{\sigma}] + i\int d\tilde{k} \int d\tilde{k}'\, \mathbf{K}(T_s, \vec{k})\mathcal{G}(\vec{k}, \vec{k}')\Delta(\vec{k}', \vec{q})},
\end{aligned}
$$

$$
N_\phi = \tilde{c}\frac{P^\tau_\phi}{m} + \int d\tilde{q} \mathcal{D}_{\vec{q}}\mathbf{H}(T_s, \vec{q}),
$$

$$
\begin{aligned}
\phi(T_s, \vec{\sigma}) &= \int d\tilde{q} \sqrt{F^\tau(q)\omega(q)P^\tau_\phi + \mathcal{D}_{\vec{q}}\mathbf{H}(T_s, \vec{q})} \\
&\quad \Big[e^{i[\omega(q)\tilde{X}^\tau_\phi + \vec{q}\cdot\vec{\sigma}] + i\int d\tilde{k} \int d\tilde{k}'\, \mathbf{K}(T_s, \vec{k})\mathcal{G}(\vec{k}, \vec{k}')\Delta(\vec{k}', \vec{q})} + \\
&\quad + e^{-i[\omega(q)\tilde{X}^\tau_\phi + \vec{q}\cdot\vec{\sigma}] - i\int d\tilde{k} \int d\tilde{k}'\, \mathbf{K}(T_s, \vec{k})\mathcal{G}(\vec{k}, \vec{k}')\Delta(\vec{k}', \vec{q})}\Big] = \\
&= 2 \int d\tilde{q}\, \mathbf{A}_{\vec{q}}(T_s; P^\tau_\phi, \mathbf{H})\, cos \Big[\vec{q}\cdot\vec{\sigma} + \mathbf{B}_{\vec{q}}(T_s; \tilde{X}^\tau_\phi, \mathbf{K}]\Big], \\
\pi(T_s, \vec{\sigma}) &= -i \int d\tilde{q} \omega(q) \sqrt{F^\tau(q)\omega(q)P^\tau_\phi + \mathcal{D}_{\vec{q}}\mathbf{H}(T_s, \vec{q})} \\
&\quad \Big[e^{i[\omega(q)\tilde{X}^\tau_\phi + \vec{q}\cdot\vec{\sigma}] + i\int d\tilde{k} \int d\tilde{k}'\, \mathbf{K}(T_s, \vec{k})\mathcal{G}(\vec{k}, \vec{k}')\Delta(\vec{k}', \vec{q})} - \\
&\quad - e^{-i[\omega(q)\tilde{X}^\tau_\phi + \vec{q}\cdot\vec{\sigma}] - i\int d\tilde{k} \int d\tilde{k}'\, \mathbf{K}(T_s, \vec{k})\mathcal{G}(\vec{k}, \vec{k}')\Delta(\vec{k}', \vec{q})}\Big] = \\
&= -2 \int d\tilde{q}\, \omega(q) \mathbf{A}_{\vec{q}}(T_s; P^\tau_\phi, \mathbf{H})\, sin \Big[\vec{q}\cdot\vec{\sigma} + \mathbf{B}_{\vec{q}}(T_s; \tilde{X}^\tau_\phi, \mathbf{K}]\Big],
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}_{\vec q}(T_s;P^\tau_\phi,\mathbf{H}] &= \sqrt{F^\tau(q)\omega(q)P^\tau_\phi+\mathcal{D}_{\vec q}\mathbf{H}(T_s,\vec q)}=\sqrt{I(T_s,\vec q)},\\
\mathbf{B}_{\vec q}(T_s;X^\tau_\phi,\mathbf{K}] &= \int d\tilde k d\tilde k'\, \mathbf{K}(T_s,\vec k)\mathcal{G}(\vec k,\vec k')\triangle(\vec k',\vec q)+\omega(q)\tilde X^\tau_\phi=\\
&= \varphi(T_s,\vec q)-\omega(q)T_s. \qquad (3.70)
\end{aligned}
$$

The Hamiltonian $H_D = M_\phi = P^\tau_\phi$ generates the following evolution in T_s ($\stackrel{\circ}{=}$ means evaluated on the equations of motion)

$$
\begin{aligned}
&\frac{\partial}{\partial T_s}\, X^\tau_\phi \stackrel{\circ}{=} \{X^\tau_\phi,P^\tau_\phi\}=-1, \qquad \Rightarrow \quad X^\tau_\phi \stackrel{\circ}{=} -T_s,\\
&\frac{\partial}{\partial T_s}\, P^\tau_\phi \stackrel{\circ}{=} 0,\\
&\frac{\partial}{\partial T_s}\, \mathbf{H}(T_s,\vec q) \stackrel{\circ}{=} 0, \qquad \frac{\partial}{\partial T_s}\, \mathbf{K}(T_s,\vec q) \stackrel{\circ}{=} 0,\\
&\Rightarrow \quad \frac{\partial}{\partial T_s}\mathbf{A}_{\vec q}(T_s;P^\tau_\phi,\mathbf{H}] = \frac{\partial}{\partial T_s}\mathbf{B}_{\vec q}(T_s;\tilde X^\tau_\phi,\mathbf{K}] \stackrel{\circ}{=} 0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial T_s}\, \phi(T_s,\vec\sigma) \stackrel{\circ}{=} -\frac{\partial}{\partial X^\tau_\phi}\, \phi(T_s,\vec\sigma)=\pi(T_s,\vec\sigma),\\
&\frac{\partial}{\partial T_s}\, \pi(T_s,\vec\sigma) \stackrel{\circ}{=} -\frac{\partial}{\partial X^\tau_\phi}\, \pi(T_s,\vec\sigma)=-[\triangle+m^2]\phi(T_s,\vec\sigma),\\
\Rightarrow \quad &(\frac{\partial^2}{\partial T_s^2}-\frac{\partial^2}{\partial\vec\sigma^2}+m^2)\phi(T_s,\vec\sigma) \stackrel{\circ}{=} 0. \qquad (3.71)
\end{aligned}
$$

Therefore, in the free case $\mathbf{H}(T_s,\vec q)$, $\mathbf{K}(T_s,\vec q)$ are constants of the motion (complete integrability and Liouville theorem for the free Klein-Gordon field). Since the canonical variable P^τ_ϕ is the Hamiltonian for the evolution in $T_s \equiv \tau$, we need the *internal* variable $X^\tau_\phi = \tau + \tilde X^\tau_\phi$ (i.e. the *internal time variable* $\tilde X^\tau_\phi$) to write Hamilton's equations $\frac{\partial}{\partial T_s}F \stackrel{\circ}{=} \{F,P^\tau_\phi\} = -\frac{\partial F}{\partial X^\tau_\phi} = -\frac{\partial F}{\partial \tilde X^\tau_\phi}$; in the free case we have $\frac{\partial}{\partial T_s} \stackrel{\circ}{=} -\frac{\partial}{\partial X^\tau_\phi}$ on $\phi(T_s,\vec\sigma)[X^\tau_\phi,P^\tau_\phi,\mathbf{H},\mathbf{K}]$ and $\pi(T_s,\vec\sigma)[X^\tau_\phi,P^\tau_\phi,\mathbf{H},\mathbf{K}]$, so that the evolution in the time $X^\tau_\phi = T_s + \tilde X^\tau_\phi$, which takes place inside the Wigner hyperplane and which can be interpreted as an evolution in the internal time $\tilde X^\tau_\phi$, is equal and opposite to the evolution in the rest-frame time T_s from a Wigner hyperplane to the next one in the free case.

By adding the two second-class constraints $X^\tau_\phi - T_s = \tilde X^\tau_\phi \approx 0$, $P^\tau_\phi - const. \approx 0$, and by going to Dirac brackets, we get the rest-frame Hamilton-Jacobi formulation corresponding to the given constant value of the total energy: the field $\phi(T_s,\vec\sigma)$, which is T_s-independent depending only upon the internal time $\tilde X^\tau_\phi$, becomes now even $\tilde X^\tau_\phi$- independent. We find in this way a symplectic subspace (spanned by the canonical variables $\mathbf{H}$, $\mathbf{K}$) of each constant energy ($P^\tau_\phi = const.$) surface of the Klein-Gordon field. *Each constant energy surface is not a symplectic manifold. However, it turns out to be the disjoint union (over $\tilde X^\tau_\phi$) of the symplectic manifolds determined by $\tilde X^\tau_\phi = const.$, $P^\tau_\phi = const.$*

While the definitions of Subsection IIIC of the three external 4- and 3-center of mass is not changed, the three internal 3-centers of mass $\vec{q}_\phi$, $\vec{R}_\phi$ and $\vec{Y}_\phi$ of Subsection IIID have to be built by using the internal Poincare' generators (3.64) in Eqs. (3.16)-(3.19). In particular, the boosts are $\vec{K}_\phi = \tilde{X}_\phi\, \vec{P}_\phi - M_\phi\, \vec{X}_\phi + \vec{K}_{S,\phi} \stackrel{def}{=} -\, M_\phi\, \vec{R}_\phi$ with $K^r_{S,\phi} = -\hat{S}^{\tau r}_\phi$. Again, $\vec{P}_\phi \approx 0$ implies $\vec{q}_\phi \approx \vec{R}_\phi \approx \vec{Y}_\phi$, the gauge fixing $\vec{q}_\phi \approx 0$ implies $\vec{K}_\phi \approx 0$, $\vec{\lambda}(\tau) = 0$, and the selection of the centroid (3.20) that coincides with the external Fokker-Pryce 4-center of inertia. The 4-momentum of the field configuration is peaked on this world-line while the canonical variables $\mathbf{H}'(\tau, \vec{q})$, $\mathbf{K}'(\tau, \vec{q})$ characterize the relative motions with respect to the *monopole* configuration of Eq.(3.69) describing the center of mass of the field configuration.

As in the N-body case ($\vec{\eta}_+ \neq \vec{q}_+$), the canonical 3-center of mass $\vec{q}_\phi$ does not coincide with $\vec{X}_\phi$, which in turn could be better defined as the *center of phase* of the field configuration. While the gauge fixing $\vec{X}_\phi \approx 0$, used to get Eqs.(3.70), implies $\vec{K}_\phi \approx \vec{K}_{S,\phi} \neq 0$, the gauge fixing $\vec{q}_\phi \approx 0$ implies $\vec{X}_\phi \approx -\vec{K}_{S,\phi}/M_\phi$ (in fact it is Møller's definition of the 3-center of energy but only in terms of the spin part of the boost).

Note that, like in Subsection IIIE for the N-body case, there should exist a canonical transformation from the canonical basis $\tilde{X}^\tau_\phi$, $M_\phi = P^\tau_\phi$, $\vec{X}_\phi$, $\vec{P}_\phi \approx 0$, $\mathbf{H}(\tau, \vec{k})$, $\mathbf{K}(\tau, \vec{k})$, to a new basis q^τ_ϕ, $M^q_\phi = \sqrt{(P^\tau_\phi)^2 - \vec{P}^2_\phi} \approx M_\phi$ (since $\{\vec{q}_\phi, P^\tau_\phi\} = \vec{P}_\phi/P^\tau_\phi \approx 0$ is only weakly zero), $\vec{q}_\phi$, $\vec{P}_\phi \approx 0$, $\mathbf{H}_q(\tau, \vec{k})$, $\mathbf{K}_q(\tau, \vec{k})$ containing relative variables with respect to the true center of mass of the field configuration

$$\begin{array}{|l|l|l|} \hline \tilde{X}_\phi & \vec{X}_\phi & \mathbf{H}(\tau, \vec{k}) \\ \hline M_\phi & \vec{P}_\phi \approx 0 & \mathbf{K}(\tau, \vec{k}) \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|l|l|l|} \hline q^\tau_\phi & \vec{q}_\phi & \mathbf{H}_q(\tau, \vec{k}) \\ \hline M^q_\phi \approx M_\phi & \vec{P}_\phi \approx 0 & \mathbf{K}_q(\tau, \vec{k}) \\ \hline \end{array} \tag{3.72}$$

It does not seem easy, however, to characterize this final canonical basis. Besides the extension of the Gartenhaus-Schwartz methods from particles to fields (so that $\mathbf{H}_q \approx \mathbf{H}$, $\mathbf{K}_q \approx \mathbf{K}$), the real problem is finding the new *internal time* variable q^τ_ϕ.

In final canonical basis we would have still Eqs.(3.70) but with $\mathbf{H}_q(\tau, \vec{k})$ and $\mathbf{K}_q(\tau, \vec{k})$ replacing $\mathbf{H}(\tau, \vec{k})$ and $\mathbf{K}(\tau, \vec{k})$ and with q^τ_ϕ replacing $\tilde{X}_\phi$ as internal time. Since both the gauge fixings $\vec{q}_\phi \approx 0$ and $\vec{X}_\phi \approx 0$ give $\vec{\lambda}(\tau) = 0$, both of them identify the same centroid (3.20) but lead to different internal times and relative variables connected by the canonical transformation (3.72).

It is under investigation [52] the problem of characterizing the configurations of the real Klein-Gordon field in terms of dynamical body frames and canonical spin bases, to the effect of finding its orientation-shape variables. Ref.[26] contains also the treatment of the coupling of the real Klein-Gordon field to scalar particles and that of the charged (complex) Klein-Gordon field to electro-magnetic field. The collective and relative variables of the electro-magnetic field are now under investigation: they could be relevant for the problem of phases in optics and laser physics [53]. Finally, Ref.[27] contains the analysis of relativistic perfect fluids along these lines.

IV. THE MULTIPOLAR EXPANSION.

In practice one is neither able to follow the motion of the particles of an open cluster in interaction with the environment nor to describe extended continuous bodies (for instance a satellite or a star), unless either the cluster or the body is approximated with a multi-polar expansion (often a pole-dipole approximation is enough).

In this Section, after the treatment of the non-relativistic case in Subsection A, we show that the rest-frame instant form provides the natural framework for studying relativistic multipolar expansions of N free particles (Subsection B), of open clusters of particles (Subsection C) and of the classical Klein-Gordon field (Subsection D) as a prototype of extended continuous systems (perfect fluids, electro-magnetic field,...).

A. The non-relativistic case.

The review paper of Ref.[29] contains the Newtonian multipolar expansions for a continuum isentropic distribution of matter characterized by a mass density $\rho(t,\vec{\sigma})$, a velocity field $U^r(t,\vec{\sigma})$, and a stress tensor $\sigma^{rs}(t,\vec{\sigma})$, with $\rho(t,\vec{\sigma})\vec{U}(t,\vec{\sigma})$ the momentum density. In case the system is isolated, the only dynamical equations are the mass conservation and the continuity equations of motion

$$\begin{aligned}
&\frac{\partial\rho(t,\vec{\sigma})}{\partial t}-\frac{\partial\rho(t,\vec{\sigma})U^r(t,\vec{\sigma})}{\partial\sigma^r}=0,\\
&\frac{\partial\rho(t,\vec{\sigma})U^r(t,\vec{\sigma})}{\partial t}-\frac{\partial[\rho U^rU^s-\sigma^{rs}](t,\vec{\sigma})}{\partial\sigma^s}\,\overset{\circ}{=}\,0,
\end{aligned}\tag{4.1}$$

respectively.

This description can be adapted to an isolated system of N particles in the following way. The mass density

$$\rho(t,\vec{\sigma})=\sum_{i=1}^N m_i\delta^3(\vec{\sigma}-\vec{\eta}_i(t)),\tag{4.2}$$

satisfies

$$\frac{\partial\rho(t,\vec{\sigma})}{\partial t}=-\sum_{i=1}^N m_i\dot{\vec{\eta}}_i(t)\cdot\vec{\partial}_{\vec{\eta}_i}\delta^3(\vec{\sigma}-\vec{\eta}_i(t))\overset{def}{=}\frac{\partial}{\partial\sigma^r}[\rho U^r](t,\vec{\sigma}),\tag{4.3}$$

while the momentum density (this can be taken as the definitory equation for the velocity field) is

$$\rho(t,\vec{\sigma})U^r(t,\vec{\sigma})=\sum_{i=1}^N m_i\dot{\vec{\eta}}_i(t)\delta^3(\vec{\sigma}-\vec{\eta}_i(t)).\tag{4.4}$$

The associated constant of motion is the total mass $m=\sum_{i=1}^N$.

Introducing a function $\zeta(\vec{\sigma},\vec{\eta}_i)$ concentrated in the N points $\vec{\eta}_i$, i=1,..,N, such that $\zeta(\vec{\sigma},\vec{\eta}_i)=0$ for $\vec{\sigma}\neq\vec{\eta}_i$ and $\zeta(\vec{\eta}_i,\vec{\eta}_j)=\delta_{ij}$ (it is a limit concept deriving from the characteristic function of a manifold), the velocity field associated to N particles becomes

$$\vec{U}(t,\vec{\sigma}) = \sum_{i=1}^{N} \dot{\vec{\eta}}_i(t)\zeta(\vec{\sigma},\vec{\eta}_i(t)). \tag{4.5}$$

The continuity equations of motion are replaced by

$$\frac{\partial}{\partial t}[\rho(t,\vec{\sigma})U^r(t,\vec{\sigma})] \overset{\circ}{=} \frac{\partial}{\partial\sigma^s}\sum_{i=1}^{N} m_i\dot{\eta}_i^r(t)\dot{\eta}_i^s(t)\delta^3(\vec{\sigma}-\vec{\eta}_i(t)) + \sum_{i=1}^{N} m_i\ddot{\eta}_i^r(t) =$$

$$\overset{def}{=} \frac{\partial[\rho U^r U^s - \sigma^{rs}](t,\vec{\sigma})}{\partial\sigma^s}. \tag{4.6}$$

For a system of free particles we have $\ddot{\vec{\eta}}_i(t) \overset{\circ}{=} 0$ so that $\sigma^{rs}(t,\vec{\sigma}) = 0$. If there are inter-particle interactions, they will determine the effective stress tensor.

Consider now an arbitrary point $\vec{\eta}(t)$. The *multipole moments* of the mass density ρ, the momentum density $\rho\vec{U}$ and the stress-like density $\rho U^r U^s$, with respect to the point $\vec{\eta}(t)$, are defined by setting ($N \geq 0$)

$$\begin{aligned}
m^{r_1...r_n}[\vec{\eta}(t)] &= \int d^3\sigma[\sigma^{r_1}-\eta^{r_1}(t)]...[\sigma^{r_n}-\eta^{r_n}(t)]\rho(\tau,\vec{\sigma}) = \\
&= \sum_{i=1}^{N} m_i[\eta_i^{r_1}(\tau)-\eta^{r_1}(t)]...[\eta_i^{r_n}(t)-\eta^{r_n}(t)], \\
&\quad n=0 \qquad m[\vec{\eta}(t)] = m = \sum_{i=1}^{N} m_i, \\
p^{r_1...r_n r}[\vec{\eta}(t)] &= \int d^3\sigma[\sigma^{r_1}-\eta^{r_1}(t)]...[\sigma^{r_n}-\eta^{r_n}(t)]\rho(t,\vec{\sigma})U^r(t,\vec{\sigma}) = \\
&= \sum_{i=1}^{N} m_i\dot{\eta}_i^r(t)[\eta_i^{r_1}(t)-\eta^{r_1}(t)]...[\eta_i^{r_n}(t)-\eta^{r_n}(t)], \\
&\quad n=0 \qquad p^r[\vec{\eta}(t)] = \sum_{i=1}^{N} m_i\dot{\eta}_i(t) = \sum_{i=1}^{N}\kappa_i^r = \kappa_+^r \approx 0, \\
p^{r_1...r_n rs}[\vec{\eta}(t)] &= \int d^3\sigma[\sigma^{r_1}-\eta^{r_1}(t)]...[\sigma^{r_n}-\eta^{r_n}(t)]\rho(t,\vec{\sigma})U^r(t,\vec{\sigma})U^s(t,\vec{\sigma}) = \\
&= \sum_{i=1}^{N} m_i\dot{\eta}_i^r(t)\dot{\eta}_i^s(t)[\eta_i^{r_1}(t)-\eta^{r_1}(t)]...[\eta_i^{r_n}(t)-\eta^{r_n}(t)].
\end{aligned} \tag{4.7}$$

The *mass monopole* is the conserved mass, while the *momentum monopole* is the total 3-momentum, vanishing in the rest frame.

If the *mass dipole* vanishes, the point $\vec{\eta}(t)$ is the *center of mass*:

$$m^r[\vec{\eta}(t)] = \sum_{i=1}^{N} m_i[\eta_i^r(t)-\eta^r(t)] = 0 \Rightarrow \vec{\eta}(t) = \vec{q}_{nr}. \tag{4.8}$$

The time derivative of the mass dipole is

$$\frac{dm^r[\vec{\eta}(t)]}{dt} = p^r[\vec{\eta}(t)] - m\dot{\eta}^r(t) = \kappa^r_+ - m\dot{\eta}^r(t). \tag{4.9}$$

When $\vec{\eta}(t) = \vec{q}_{nr}$, from the vanishing of this time derivative we get the *momentum-velocity relation for the center of mass*

$$p^r[\vec{q}_{nr}] = \kappa^r_+ = m\dot{q}^r_+ \quad [\approx 0 \;\; in\, the\, rest\, frame]. \tag{4.10}$$

The *mass quadrupole* is

$$m^{rs}[\vec{\eta}(t)] = \sum_{i=1}^N m_i\eta^r_i(t)\eta^s_i(t) - m\eta^r(t)\eta^s(t) - \Big(\eta^r(t)m^s[\vec{\eta}(t)] + \eta^s(t)m^r[\vec{\eta}(t)]\Big), \tag{4.11}$$

so that the *barycentric mass quadrupole and tensor of inertia* are, respectively

$$\begin{aligned}
m^{rs}[\vec{q}_{nr}] &= \sum_{i=1}^N m_i\eta^r_i(t)\eta^s_i(t) - mq^r_{nr}q^s_{nr},\\
I^{rs}[\vec{q}_{nr}] &= \delta^{rs}\sum_u m^{uu}[\vec{q}_{nr}] - m^{rs}[\vec{q}_{nr}] =\\
&= \sum_{i=1} m_i[\delta^{rs}\vec{\eta}^2_i(t) - \eta^r_i(t)\eta^s_i(t)] - m[\delta^{rs}\vec{q}^2_{nr} - q^r_{nr}q^s_{nr}] =\\
&= \sum_{a,b}^{1...N-1} k_{ab}(\vec{\rho}_a\cdot\vec{\rho}_b\delta^{rs} - \rho^r_a\rho^s_b),\\
\Rightarrow \qquad & m^{rs}[\vec{q}_{nr}] = \delta^{rs}\sum_{a,b=1}^{N-1} k_{ab}\,\vec{\rho}_a\cdot\vec{\rho}_b - I^{rs}[\vec{q}_{nr}].
\end{aligned} \tag{4.12}$$

The antisymmetric part of the barycentric momentum dipole gives rise to the *spin vector* in the following way

$$\begin{aligned}
p^{rs}[\vec{q}_{nr}] &= \sum_{i=1}^N m_i\eta^r_i(t)\dot{\eta}^s_i(t) - q^r_{nr}p^s[\vec{q}_{nr}] = \sum_{i=1}^N \eta^r_i(t)\kappa^s_i(t) - q^r_+\kappa^s_+,\\
S^u &= \frac{1}{2}\epsilon^{urs}p^{rs}[\vec{q}_{nr}] = \sum_{a=1}^{N-1}(\vec{\rho}_a\times\vec{\pi}_{qa})^u.
\end{aligned} \tag{4.13}$$

The *multipolar expansions* of the mass and momentum densities around the point $\vec{\eta}(t)$ are

$$\begin{aligned}
\rho(t,\vec{\sigma}) &= \sum_{n=0}^\infty \frac{m^{r_1...r_n}[\vec{\eta}]}{n!}\frac{\partial^n}{\partial\sigma^{r_1}...\partial\sigma^{r_n}}\delta^3(\vec{\sigma}-\vec{\eta}(t)),\\
\rho(t,\vec{\sigma})U^r(t,\vec{\sigma}) &= \sum_{n=0}^\infty \frac{p^{r_1...r_nr}[\vec{\eta}]}{n!}\frac{\partial^n}{\partial\sigma^{r_1}...\partial\sigma^{r_n}}\delta^3(\vec{\sigma}-\vec{\eta}(t)).
\end{aligned} \tag{4.14}$$

Finally, we get the *barycentric multipolar expansions* as

$$\begin{aligned}
\rho(t,\vec{\sigma}) &= m\delta^3(\vec{\sigma}-\vec{q}_{nr}) - \frac{1}{2}(I^{rs}[\vec{q}_{nr}] - \frac{1}{2}\delta^{rs}\sum_u I^{uu}[\vec{q}_{nr}])\frac{\partial^2}{\partial\sigma^r\partial\sigma^s}\delta^3(\vec{\sigma}-\vec{q}_{nr}) + \\
&+ \sum_{n=3}^{\infty}\frac{m^{r_1\dots r_n}[\vec{q}_{nr}]}{n!}\frac{\partial^n}{\partial\sigma^{r_1}\dots\partial\sigma^{r_n}}\delta^3(\vec{\sigma}-\vec{q}_{nr}),\\
\rho(t,\vec{\sigma})U^r(t,\vec{\sigma}) &= \kappa^r_+\delta^3(\vec{\sigma}-\vec{q}_{nr}) + \Big[\frac{1}{2}\epsilon^{rsu}S^u + p^{(sr)}[\vec{q}_{nr}]\Big]\frac{\partial}{\partial\sigma^s}\delta^3(\vec{\sigma}-\vec{q}_{nr}) + \\
&+ \sum_{n=2}^{\infty}\frac{p^{r_1\dots r_n r}[\vec{q}_{nr}]}{n!}\frac{\partial^n}{\partial\sigma^{r_1}\dots\partial\sigma^{r_n}}\delta^3(\vec{\sigma}-\vec{q}_{nr}).
\end{aligned} \tag{4.15}$$

B. The relativistic case on a Wigner hyper-plane.

It is shown in Ref.[12] that, on a Wigner hyperplane with $\tau \equiv T_s$, the energy-momentum of N free scalar particles has the form

$$\begin{aligned}
T^{\mu\nu}[x^\beta_s(T_s) + \epsilon^\beta_u(u(p_s))\sigma^u] &= \epsilon^\mu_A(u(p_s))\epsilon^\nu_B(u(p_s))T^{AB}(T_s,\vec{\sigma}) = \\
&= \sum_{i=1}^N \delta^3(\vec{\sigma}-\vec{\eta}_i(T_s))\Big[\sqrt{m_i^2+\vec{\kappa}_i^2(T_s)}\,u^\mu(p_s)u^\nu(p_s) + \\
&+ k^r_i(T_s)\Big(u^\mu(p_s)\epsilon^\nu_r(u(p_s)) + u^\nu(p_s)\epsilon^\mu_r(u(p_s))\Big) + \\
&+ \frac{\kappa^r_i(T_s)\kappa^s_i(T_s)}{\sqrt{m_i^2+\vec{\kappa}_i^2(T_s)}}\epsilon^\mu_r(u(p_s))\epsilon^\nu_s(u(p_s))\Big],\\
T^{\tau\tau}(T_s,\vec{\sigma}) &= \sum_{i=1}^N \delta^3(\vec{\sigma}-\vec{\eta}_i(T_s))\sqrt{m_i^2+\vec{\kappa}_i^2(T_s)}),\\
T^{\tau r}(T_s,\vec{\sigma}) &= \sum_{i=1}^N \delta^3(\vec{\sigma}-\vec{\eta}_i(T_s))\kappa^r_i(T_s),\\
T^{rs}(T_s,\vec{\sigma}) &= \sum_{i=1}^N \delta^3(\vec{\sigma}-\vec{\eta}_i(T_s))\frac{\kappa^r_i(T_s)\kappa^s_i(T_s)}{\sqrt{m_i^2+\vec{\kappa}_i^2(T_s)}},\\
P^\mu_T &= p^\mu_s = M_{sys}\,u^\mu(p_s) + \epsilon^\mu_r(u(p_s))\kappa^r_+ \approx M_{sys}\,u^\mu(p_s),\\
M_{sys} &= \sum_{i=1}^N \sqrt{m_i^2c^2+\vec{\kappa}_i^2(T_s)}.
\end{aligned} \tag{4.16}$$

We will now define the special relativistic Dixon multi-poles on the Wigner hyperplane with $T_s - \tau \equiv 0$ for the N-body problem (see the bibliography of Ref.[11] for older attempts). Note that, since we have not yet added the gauge fixings $\vec{q}_+ \approx 0$, the centroid origin of the 3-coordinates has the form $x^\mu_s(\tau) = x_s^{(\vec{q}_+)\mu}(\tau) + \int_o^\tau d\tau_1\,\lambda_r(\tau_1)$ according to Eqs.(3.7) and (3.20).

Consider an arbitrary time-like world-line $w^\mu(\tau) = z^\mu(\tau, \vec{\eta}(\tau)) = x^\mu_s(\tau) + \epsilon^\mu_r(u(p_s))\, \eta^r(\tau) = x^{(\vec{q}_+)\mu}_s(\tau) + \epsilon^\mu_r(u(p_s))\, \tilde{\eta}^r(\tau)$ $[\tilde{\eta}^r(\tau) = \eta^r(\tau) + \int_o^\tau d\tau_1\, \lambda_r(\tau_1)]$ and evaluate the Dixon multi-poles [28] on the Wigner hyper-planes in the natural gauge with respect to the given world-line. A generic point will be parametrized by

$$
\begin{aligned}
z^\mu(\tau, \vec{\sigma}) &= x^\mu_s(\tau) + \epsilon^\mu_r(u(p_s))\, \sigma^r = \\
&= w^\mu(\tau) + \epsilon^\mu_r(u(p_s))[\sigma^r - \eta^r(\tau)] \stackrel{def}{=} w^\mu(\tau) + \delta z^\mu(\tau, \vec{\sigma}),
\end{aligned}
\tag{4.17}
$$

so that $\delta z_\mu(\tau, \vec{\sigma}) u^\mu(p_s) = 0$.

While for $\tilde{\vec{\eta}}(\tau) = 0$ $[\vec{\eta}(\tau) = \int_o^\tau d\tau_1\, \lambda_r(\tau_1)]$ we get the multi-poles relative to the centroid $x^\mu_s(\tau)$, for $\vec{\eta}(\tau) = 0$ we get those relative to the centroid $x^{(\vec{q}_+)\mu}_s(\tau)$. In the gauge $\vec{R}_+ \approx \vec{q}_+ \approx \vec{y}_+ \approx 0$, where $\vec{\lambda}(\tau) = 0$, it follows that $\vec{\eta}(\tau) = \tilde{\vec{\eta}}(\tau) = 0$ identifies the *barycentric* multi-poles with respect to the centroid $x^{(\vec{q}_+)\mu}_s(\tau)$, that now carries the internal 3-center of mass.

Lorentz covariant *Dixon's multi-poles* and their Wigner covariant counterparts on the Wigner hyper-planes are then defined as

$$
\begin{aligned}
t^{\mu_1 \dots \mu_n \mu\nu}_T(T_s, \vec{\eta}) &= t^{(\mu_1 \dots \mu_n)(\mu\nu)}_T(T_s, \vec{\eta}) = \\
&= \epsilon^{\mu_1}_{r_1}(u(p_s)) \dots \epsilon^{\mu_n}_{r_n}(u(p_s))\, \epsilon^\mu_A(u(p_s)) \epsilon^\nu_B(u(p_s)) q^{r_1 .. r_n AB}_T(T_s, \vec{\eta}) = \\
&= \int d^3\sigma \delta z^{\mu_1}(T_s, \vec{\sigma}) \dots \delta z^{\mu_n}(T_s, \vec{\sigma}) T^{\mu\nu}[x^{(\vec{q}_+)\beta}_s(T_s) + \epsilon^\beta_u(u(p_s))\sigma^u] = \\
&= \epsilon^\mu_A(u(p_s)) \epsilon^\nu_B(u(p_s)) \int d^3\sigma \delta z^{\mu_1}(T_s, \vec{\sigma}) \dots. \delta z^{\mu_n}(T_s, \vec{\sigma}) T^{AB}(T_s, \vec{\sigma}) = \\
&= \epsilon^{\mu_1}_{r_1}(u(p_s)) \dots \epsilon^{\mu_n}_{r_n}(u(p_s)) \\
&\quad \Big[u^\mu(p_s) u^\nu(p_s) \sum_{i=1}^N [\eta^{r_1}_i(T_s) - \eta^{r_1}(T_s)] \dots [\eta^{r_n}_i(T_s) - \eta^{r_n}(T_s)] \sqrt{m_i^2 + \vec{\kappa}_i^2(T_s)} + \\
&+ \epsilon^\mu_r(u(p_s)) \epsilon^\nu_s(u(p_s)) \\
&\quad \sum_{i=1}^N [\eta^{r_1}_i(T_s) - \eta^{r_1}(T_s)] \dots [\eta^{r_n}_i(T_s) - \eta^{r_n}(T_s)] \frac{\kappa^r_i(T_s) \kappa^s_i(T_s)}{\sqrt{m_i^2 + \vec{\kappa}_i^2(T_s)}} + \\
&+ [u^\mu(p_s) \epsilon^\nu_r(u(p_s)) + u^\nu(p_s) \epsilon^\mu_r(u(p_s))] \\
&\quad \sum_{i=1}^N [\eta^{r_1}_i(T_s) - \eta^{r_1}(T_s)] \dots [\eta^{r_n}_i(T_s) - \eta^{r_n}(T_s)] \kappa^r_i(T_s) \Big],
\end{aligned}
$$

$$
q^{r_1 \dots r_n AB}_T(T_s, \vec{\eta}) = \int d^3\sigma\, [\sigma^{r_1} - \eta^{r_1}(T_s)] \dots [\sigma^{r_n} - \eta^{r_n}]\, T^{AB}(T_s, \vec{\sigma}) =
$$

$$= \delta^A_\tau \delta^B_\tau \sum_{i=1}^N [\eta_i^{r_1}(T_s) - \eta^{r_1}(T_s)]...[\eta_i^{r_n}(T_s) - \eta^{r_n}(T_s)] \sqrt{m_i^2 + \vec{\kappa}_i^2(T_s)} +$$

$$+ \delta^A_u \delta^B_v \sum_{i=1}^N [\eta_i^{r_1}(T_s) - \eta^{r_1}(T_s)]...[\eta_i^{r_n}(T_s) - \eta^{r_n}(T_s)] \frac{\kappa_i^u(T_s)\kappa_i^v(T_s)}{\sqrt{m_i^2 + \vec{\kappa}_i^2(T_s)}} +$$

$$+ (\delta^A_\tau \delta^B_u + \delta^A_u \delta^B_\tau) \sum_{i=1}^N [\eta_i^{r_1}(T_s) - \eta^{r_1}(T_s)]...[\eta_i^{r_n}(T_s) - \eta^{r_n}(T_s)] \kappa_i^r(T_s),$$

$$u_{\mu_1}(p_s) \qquad t_T^{\mu_1...\mu_n\mu\nu}(T_s, \vec{\eta}) = 0. \tag{4.18}$$

Related multi-poles are $p_T^{\mu_1..\mu_n\mu}(T_s, \vec{\eta}) = t_T^{\mu_1...\mu_n\mu\nu}(T_s, \vec{\eta})\, u_\nu(p_s) = \epsilon^{\mu_1}_{r_1}(u(p_s))...\epsilon^{\mu_n}_{r_n}(u(p_s))\epsilon^\mu_A(u(p_s))\, q_T^{r_1...r_n A\tau}(T_s, \vec{\eta})$. They satisfy $u_{\mu_1}(p_s) p_T^{\mu_1...\mu_n\mu}(T_s, \vec{\eta}) = 0$ and for $n = 0$ they imply $p_T^\mu(T_s, \vec{\eta}) = \epsilon^\mu_A(u(p_s)) q_T^{A\tau}(T_s) = P_T^\mu \approx p_s^\mu$.

The inverse formulas, giving the *multipolar expansion*, are

$$T^{\mu\nu}[w^\beta(T_s) + \delta z^\beta(T_s, \vec{\sigma})] = T^{\mu\nu}[x_s^{(\vec{q}_+)\beta}(T_s) + \epsilon^\beta_r(u(p_s))\, \sigma^r] =$$

$$= \epsilon^\mu_A(u(p_s))\epsilon^\nu_B(u(p_s)) T^{AB}(T_s, \vec{\sigma}) =$$

$$= \epsilon^\mu_A(u(p_s))\epsilon^\nu_B(u(p_s)) \sum_{n=0}^\infty (-1)^n \frac{q_T^{r_1...r_n AB}(T_s, \vec{\eta})}{n!} \qquad \frac{\partial^n}{\partial \sigma^{r_1}...\partial \sigma^{r_n}} \delta^3(\vec{\sigma} - \vec{\eta}(T_s)). \tag{4.19}$$

Note however that, as pointed out by Dixon [28], the distributional equation (4.19) is valid only if analytic test functions are used defined on the support of the energy-momentum tensor.

The quantities $q_T^{r_1...r_n\tau\tau}(T_s, \vec{\eta})$, $q_T^{r_1...r_n r\tau}(T_s, \vec{\eta}) = q_T^{r_1...r_n\tau r}(T_s, \vec{\eta})$, $q_T^{r_1...r_n uv}(T_s, \vec{\eta})$ are the *mass density, momentum density and stress tensor multi-poles* with respect to the world-line $w^\mu(T_s)$ (barycentric for $\vec{\eta} = \dot{\vec{\eta}} = 0$).

1. Monopoles

The *monopoles*, corresponding to $n = 0$, have the following expression (they are $\vec{\eta}$-independent)

$$q_T^{AB}(T_s, \vec{\eta}) = \delta^A_\tau \delta^B_\tau M + \delta^A_u \delta^B_v \sum_{i=1}^N \frac{\kappa_i^u \kappa_i^v}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} + (\delta^A_\tau \delta^B_u + \delta^A_u \delta^B_\tau) \kappa_+^u,$$

$$
\begin{aligned}
q_T^{\tau\tau}(T_s, \vec{\eta}) &\rightarrow_{c\rightarrow\infty} \sum_{i=1}^N m_i c^2 + H_{rel} + O(1/c),\\
q_T^{r\tau}(T_s, \vec{\eta}) &= \kappa_+^r \approx 0, \quad rest-frame\, condition\, (also\, at\, the\, non-relativistic\, level),\\
q_T^{uv}(T_s, \vec{\eta}) &\rightarrow_{c\rightarrow\infty} \sum_{ab}^{1..N-1} k_{ab}^{-1} \pi_{qa}^u \pi_{qb}^v + O(1/c),\\
q_{TA}^A(T_s, \vec{\eta}) &= t_{T\mu}^\mu(T_s, \vec{\eta}) = \sum_{i=1}^N \frac{m_i^2}{\sqrt{m_i^2 + \vec{\kappa}_i^2}}\\
&\rightarrow_{c\rightarrow\infty} \sum_{i=1}^N m_i c^2 - H_{rel} + O(1/c). \qquad (4.20)
\end{aligned}
$$

Therefore, independently of the choice of the world-line $w^\mu(\tau)$, in the rest-frame instant form the *mass monopole* $q_T^{\tau\tau}$ is the invariant mass $M_{sys} = \sum_{i=1}^N \sqrt{m_i^2 + \vec{\kappa}_i^2}$, while the *momentum monopole* $q_T^{r\tau}$ vanishes and q_T^{uv} is the *stress tensor monopole*.

2. Dipoles

The mass, momentum and stress tensor *dipoles*, corresponding to $n = 1$, are

$$
\begin{aligned}
q_T^{rAB}(T_s, \vec{\eta}) \,&=\, \delta_\tau^A \delta_\tau^B M [R_+^r(T_s) - \eta^r(T_s)] + \delta_u^A \delta_v^B \Big[\sum_{i=1}^N \frac{\eta_i^r \kappa_i^u \kappa_i^v}{\sqrt{m_i^2 + \vec{\kappa}_i^2}}(T_s) - \eta^r(T_s) q_T^{uv}(T_s, \vec{\eta}) \Big] +\\
&+\, (\delta_\tau^A \delta_u^B + \delta_u^A \delta_\tau^B) \Big[\sum_{i=1}^N [\eta_i^r \kappa_i^u](T_s) - \eta^r(T_s) \kappa_+^u \Big]. \qquad (4.21)
\end{aligned}
$$

The vanishing of the *mass dipole* $q_T^{r\tau\tau}$ implies $\vec{\eta}(\tau) = \tilde{\vec{\eta}}(\tau) - \int_o^\tau d\tau_1\, \vec{\lambda}(\tau_1) = \vec{R}_+$ and identifies the world-line $w^\mu(\tau) = x_s^{(\vec{q}_+)\mu}(\tau) + \epsilon_r^\mu(u(p_s)) \Big[R_+^r + \int_o^\tau d\tau_1\, \lambda_r(\tau_1) \Big]$. In the gauge $\vec{R}_+ \approx \vec{q}_+ \approx \vec{y}_+ \approx 0$, where $\vec{\lambda}(\tau) = 0$, this is the world-line $w^\mu(\tau) = x_s^{(\vec{q}_+)\mu}(\tau)$ of the centroid associated with the *rest-frame internal 3-center of mass* $\vec{q}_+$. We have, therefore, the implications following from the vanishing of the barycentric (i.e. $\vec{\lambda}(\tau) = 0$) mass dipole

$$
\begin{aligned}
q_T^{r\tau\tau}(T_s, \vec{\eta}) \,&=\, \epsilon_{\mu_1}^{r_1}(u(p_s)) \tilde{t}_T^{\mu_1}(T_s, \vec{\eta}) = M \Big[R_+^r(T_s) - \eta^r(T_s) \Big] = 0, \quad and\ \vec{\lambda}(\tau) = 0,\\
&\Rightarrow \quad \vec{\eta}(T_s) = \tilde{\vec{\eta}}(T_s) = \vec{R}_+ \approx \vec{q}_+ \approx \vec{y}_+. \qquad (4.22)
\end{aligned}
$$

In the gauge $\vec{R}_+ \approx \vec{q}_+ \approx \vec{y}_+ \approx 0$, Eq.(4.22) with $\vec{\eta} = \vec{\tilde{\eta}} = 0$ implies the vanishing of the time derivative of the barycentric mass dipole: this identifies the *center-of-mass momentum-velocity relation* (or constitutive equation) for the system

$$\frac{dq_T^{r\tau\tau}(T_s, \vec{\eta})}{dT_s} \stackrel{\circ}{=} \kappa_+^r - M\dot{R}_+^r = 0. \tag{4.23}$$

The expression of the barycentric dipoles in terms of the internal relative variables, when $\vec{\eta} = \vec{\tilde{\eta}} = \vec{R}_+ \approx \vec{q}_+ \approx 0$ and $\vec{\kappa}_+ \approx 0$, is obtained by using the Gartenhaus-Schwartz transformation.

$$\begin{aligned}
q_T^{r\tau\tau}(T_s, \vec{R}_+) &= 0, \\
q_T^{ru\tau}(T_s, \vec{R}_+) &= \sum_{i=1}^{N} \eta_i^r \kappa_i^u - R_+^r \kappa_+^u = \sum_{a=1}^{N-1} \rho_a^r \pi_a^u + (\eta_+^r - R_+^r)\kappa_+^u \\
&\rightarrow_{c\rightarrow\infty} \sum_{a=1}^{N-1} \rho_a^r \pi_{qa}^u = \sum_{ab}^{1..N-1} k_{ab} \rho_a^r \dot{\rho}_b^u, \\
q_T^{ruv}(T_s, \vec{R}_+) &= \sum_{i=1}^{N} \eta_i^r \frac{\kappa_i^u \kappa_i^v}{H_i} - R_+^r \sum_{i=1}^{N} \frac{\kappa_i^u \kappa_i^v}{H_i} = \\
&= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \sum_{a=1}^{N-1} \gamma_{ai} \rho_a^r \frac{\kappa_i^u \kappa_i^v}{H_i} + (\eta_+^r - R_+^r) \sum_{i=1}^{N} \frac{\kappa_i^u \kappa_i^v}{H_i} \\
&\rightarrow_{c\rightarrow\infty} \frac{1}{\sqrt{N}} \sum_{abc}^{1..N-1} \Big[N \sum_{i=1}^{N} \frac{\gamma_{ai}\gamma_{bi}\gamma_{ci}}{m_i} - \frac{\sum_{j=1}^{N} m_j \gamma_{aj}}{m} \Big] \rho_a^r \pi_{qb}^u \pi_{qc}^v + O(1/c).
\end{aligned} \tag{4.24}$$

The antisymmetric part of the related dipole $p_T^{\mu_1\mu}(T_s, \vec{\eta})$ identifies the *spin tensor*. Indeed, the *spin dipole* is

$$\begin{aligned}
S_T^{\mu\nu}(T_s)[\vec{\eta}] &= 2p_T^{[\mu\nu]}(T_s, \vec{\eta}) = 2\epsilon_r^{[\mu}(u(p_s))\, \epsilon_A^{\nu]}(u(p_s))\, q_T^{rA\tau}(T_s, \vec{\eta}) = \\
&= M_{sys}\, [R_+^r(T_s) - \eta^r(T_s)] \Big[\epsilon_r^\mu(u(p_s)) u^\nu(p_s) - \epsilon_r^\nu(u(p_s)) u^\mu(p_s) \Big] + \\
&+ \sum_{i=1}^{N} [\eta_i^r(T_s) - \eta^r(T_s)] \kappa_i^s(T_s) \Big[\epsilon_r^\mu(u(p_s)) \epsilon_s^\nu(u(p_s)) - \epsilon_r^\nu(u(p_s)) \epsilon_s^\mu(u(p_s)) \Big], \\
m_{u(p_s)}^\nu(T_s, \vec{\eta}) &= u_\mu(p_s) S_T^{\mu\nu}(T_s)[\vec{\eta}] = -\epsilon_r^\nu(u(p_s)) [\bar{S}_s^{\tau r} - M_{sys}\, \eta^r(T_s)] = \\
&= -\epsilon_r^\nu(u(p_s)) M_{sys}\, [R_+^r(T_s) - \eta^r(T_s)] = -\epsilon_r^\nu(u(p_s)) q_T^{r\tau\tau}(T_s, \vec{\eta}), \\
\Rightarrow \quad & u_\mu(p_s) S_T^{\mu\nu}(T_s)[\vec{\eta}] = 0, \qquad \Rightarrow \vec{\eta} = \vec{R}_+,
\end{aligned}$$

$$\Downarrow \quad barycentric\, spin\ for\ \vec{\kappa}_+ \approx 0,\ \vec{\eta} = \vec{\tilde{\eta}} = 0,\ see\, Eq(4.17),$$

$$S^{\mu\nu}_T(T_s)[\vec{\eta} = 0] \;=\; S^{\mu\nu}_s \stackrel{\circ}{=} \epsilon^{rsu}\, \bar{S}^u_s\, \epsilon^\mu_r(u(p_s))\epsilon^\nu_s(u(p_s)). \tag{4.25}$$

This explains why $m^\mu_{u(p_s)}(T_s, \vec{\eta})$ is also called the *mass dipole moment*.

We find, therefore, that in the gauge $\vec{R}_+ \approx \vec{q}_+ \approx \vec{y}_+ \approx 0$, with $P^\mu_T \approx M_{sys}\, u^\mu(p_s) = M_{sys}\, \dot{x}^{(\vec{q}_+)\mu}_s(T_s)$, the Møller and barycentric centroid $x^{(\vec{q}_+)\mu}_s(T_s)$ is simultaneously the *Tulczyjew centroid* [48] (defined by $S^{\mu\nu}\, P_\nu = 0$) and also the *Pirani centroid* [47] (defined by $S^{\mu\nu}\, \dot{x}^{(\vec{q}_+)}_{s\nu} = 0$). In general, *lacking a relation between 4-momentum and 4-velocity*, they are different centroids.

3. Quadrupoles and the barycentric tensor of inertia

The *quadrupoles*, corresponding to $n = 2$, are

$$\begin{aligned}
q^{r_1r_2AB}_T(T_s, \vec{\eta}) \;&=\; \delta^A_\tau\delta^B_\tau \sum_{i=1}^N [\eta^{r_1}_i(T_s) - \eta^{r_1}(T_s)][\eta^{r_2}_i(T_s) - \eta^{r_2}(T_s)]\sqrt{m^2_i + \vec{\kappa}^2_i(T_s)} + \\
&+\; \delta^A_u\delta^B_v \sum_{i=1}^N [\eta^{r_1}_i(T_s) - \eta^{r_1}(T_s)][\eta^{r_2}_i(T_s) - \eta^{r_2}(T_s)] \frac{\kappa^u_i\kappa^v_i}{\sqrt{m^2_i + \vec{\kappa}^2_i}}(T_s) + \\
&+\; (\delta^A_\tau\delta^B_u + \delta^A_u\delta^B_\tau) \sum_{i=1}^N [\eta^{r_1}_i(T_s) - \eta^{r_1}(T_s)][\eta^{r_2}_i(T_s) - \eta^{r_2}(T_s)]\kappa^u_i(T_s), \\
&.
\end{aligned} \tag{4.26}$$

When the mass dipole vanishes, i.e. $\vec{\eta} = \vec{R}_+ = \sum_i \vec{\eta}_i \sqrt{m^2_i + \vec{\kappa}^2_i}/M_{sys}$, we get

$$\begin{aligned}
q^{r_1r_2\tau\tau}_T(T_s, \vec{R}_+) \;&=\; \sum_{i=1}^N (\eta^{r_1}_i - R^{r_1}_+)(\eta^{r_2}_i - R^{r_2}_+)\sqrt{m^2_i + \vec{\kappa}^2_i(T_s)}, \\
q^{r_1r_2u\tau}_T(T_s, \vec{R}_+) \;&=\; \sum_{i=1}^N (\eta^{r_1}_i - R^{r_1}_+)(\eta^{r_2}_i - R^{r_2}_+)\kappa^u_i, \\
q^{r_1r_2uv}_T(T_s, \vec{R}_+) \;&=\; \sum_{i=1}^N (\eta^{r_1}_i - R^{r_1}_+)(\eta^{r_2}_i - R^{r_2}_+)\frac{\kappa^u_i\kappa^v_i}{\sqrt{m^2_i + \vec{\kappa}^2_i(T_s)}} \\
&=\; \frac{1}{N}\sum_{ijk}^{1..N}\sum_{ab}^{1..N-1} (\gamma_{ai} - \gamma_{aj}).
\end{aligned} \tag{4.27}$$

Following the non-relativistic pattern, Dixon starts from the *mass quadrupole*

$$q_T^{r_1 r_2 \tau\tau}(T_s, \vec{R}_+) = \sum_{i=1}^{N} [\eta_i^{r_1} \eta_i^{r_2} \sqrt{m_i^2 + \vec{\kappa}_i^2}](T_s) - M_{sys}\, R_+^{r_1}\, R_+^{r_2}, \tag{4.28}$$

and defines the following *barycentric tensor of inertia*

$$\begin{aligned} I^{r_1 r_2}_{dixon}(T_s) &= \delta^{r_1 r_2} \sum_u q_T^{uu\tau\tau}(T_s, \vec{R}_+) - q_T^{r_1 r_2 \tau\tau}(T_s, \vec{R}_+) = \\ &= \sum_{i=1}^{N} [(\delta^{r_1 r_2} (\vec{\eta}_i - \vec{R}_+)^2 - (\eta_i^{r_1} - R_+^{r_1})(\eta_i^{r_2} - R_+^{r_2})) \sqrt{m_i^2 + \vec{\kappa}_i^2}](T_s) \\ &\rightarrow_{c\rightarrow\infty} \sum_{ab}^{1..N-1} k_{ab} [\vec{\rho}_{qa} \cdot \vec{\rho}_{qb} \delta^{r_1 r_2} - \rho_{qa}^{r_1} \rho_{qb}^{r_2}] + O(1/c) = I^{r_1 r_2}[\vec{q}_{nr}] + O(1/c). \end{aligned} \tag{4.29}$$

Note that in the non-relativistic limit we recover the *tensor of inertia* of Eqs.(4.11).

On the other hand, Thorne's definition of *barycentric tensor of inertia* [30] is

$$\begin{aligned} I^{r_1 r_2}_{thorne}(T_s) &= \delta^{r_1 r_2} \sum_u q_T^{uuA}{}_A(T_s, \vec{R}_+) - q_T^{r_1 r_2 A}{}_A(T_s, \vec{R}_+) = \\ &= \sum_{i=1}^{N} \frac{m_i^2 (\delta^{r_1 r_2} (\vec{\eta}_i - \vec{R}_+)^2 - (\eta_i^{r_1} - R_+^{r_1})(\eta_i^{r_2} - R_+^{r_2}))}{\sqrt{m_i^2 + \vec{\kappa}_i^2}}(T_s) \\ &\rightarrow_{c\rightarrow\infty} \sum_{ab}^{1..N-1} k_{ab} [\vec{\rho}_{qa} \cdot \vec{\rho}_{qb} \delta^{r_1 r_2} - \rho_{qa}^{r_1} \rho_{qb}^{r_2}] + O(1/c) = I^{r_1 r_2}[\vec{q}_{nr}] + O(1/c). \end{aligned} \tag{4.30}$$

In this case too we recover the *tensor of inertia* of Eq.(4.11). Note that the Dixon and Thorne barycentric tensors of inertia differ at the post-Newtonian level

$$\begin{aligned} I^{r_1 r_2}_{dixon}(T_s) - I^{r_1 r_2}_{thorne}(T_s) &= \frac{1}{c} \sum_{ab}^{1..N-1} \sum_{ijk}^{1..N} \frac{m_j m_k}{N m^2} (\gamma_{ai} - \gamma_{aj})(\gamma_{bi} - \gamma_{bk}) \\ &\Big[\vec{\rho}_{qa} \cdot \vec{\rho}_{qb} \delta^{r_1 r_2} - \rho_{qa}^{r_1} \rho_{qb}^{r_2}\Big] \frac{N \sum_{cd}^{1..N-1} \gamma_{ci} \gamma_{di} \vec{\pi}_{qc} \cdot \vec{\pi}_{qd}}{m_i} + O(1/c^2). \end{aligned} \tag{4.31}$$

4. The multipolar expansion

It can be shown that the multipolar expansion can be rearranged in the form

$$T^{\mu\nu}[x_s^{(\vec{q}_+)\beta}(T_s) + \epsilon_r^\beta(u(p_s))\sigma^r] = T^{\mu\nu}[w^\beta(T_s) + \epsilon_r^\beta(u(p_s))(\sigma^r - \eta^r(T_s))] =$$

$$\begin{aligned}
&= u^{(\mu}(p_s)\epsilon_A^{\nu)}(u(p_s))[\delta_\tau^A M_{sys} + \delta_u^A \kappa_+^u]\delta^3(\vec{\sigma} - \vec{\eta}(T_s)) + \\
&+ \frac{1}{2} S_T^{\rho(\mu}(T_s)[\vec{\eta}]u^{\nu)}(p_s)\epsilon_\rho^r(u(p_s))\frac{\partial}{\partial\sigma^r}\delta^3(\vec{\sigma} - \vec{\eta}(T_s)) + \\
&+ \sum_{n=2}^\infty \frac{(-1)^n}{n!} I_T^{\mu_1..\mu_n\mu\nu}(T_s, \vec{\eta})\epsilon_{\mu_1}^{r_1}(u(p_s))..\epsilon_{\mu_n}^{r_n}(u(p_s))\frac{\partial^n}{\partial\sigma^{r_1}..\partial\sigma^{r_n}}\delta^3(\vec{\sigma} - \vec{\eta}(T_s)), \qquad (4.32)
\end{aligned}$$

where for $n \geq 2$ and $\vec{\eta} = 0$, we have $I_T^{\mu_1..\mu_n\mu\nu}(T_s) = \frac{4(n-1)}{n+1} J_T^{(\mu_1..\mu_{n-1}|\mu|\mu_n)\nu}(T_s)$, with $J_T^{\mu_1..\mu_n\mu\nu\rho\sigma}(T_s)$ being the generalized Dixon 2^{2+n}*-pole inertial moment tensors* given in Ref.[11].

Note that, for an isolated system described by the multi-poles appearing in Eq.(4.32) [this is not true for those in Eq.(4.19)] the equations $\partial_\mu T^{\mu\nu} \circeq 0$ imply no more than the following *Papapetrou-Dixon-Souriau equations of motion* [29] for the total momentum $P_T^\mu(T_s) = \epsilon_A^\mu(u(p_s))q_T^{A\tau}(T_s) = p_s^\mu$ and the spin tensor $S_T^{\mu\nu}(T_s)[\vec{\eta} = 0]$

$$\begin{aligned}
\frac{dP_T^\mu(T_s)}{dT_s} &\circeq 0, \\
\frac{dS_T^{\mu\nu}(T_s)[\vec{\eta} = 0]}{dT_s} &\circeq 2P_T^{[\mu}(T_s)u^{\nu]}(p_s) = 2\kappa_+^u\epsilon_u^{[\mu}(u(p_s))u^{\nu]}(p_s) \approx 0,
\end{aligned}$$

$$or \qquad \frac{dM_{sys}}{dT_s} \circeq 0, \qquad \frac{d\vec{\kappa}_+}{dT_s} \circeq 0, \qquad \frac{dS_s^{\mu\nu}}{dT_s} \circeq 0. \qquad (4.33)$$

5. Cartesian Tensors

In the applications to gravitational radiation, *irreducible symmetric trace-free Cartesian tensors (STF tensors)* [30, 31] are needed instead of *Cartesian tensors*. While a *Cartesian multi-pole tensor of rank l* (like the rest-frame Dixon multi-poles) on R^3 has 3^l components, $\frac{1}{2}(l+1)(l+2)$ of which are in general independent, a *spherical multi-pole moment of order l* has only $2l+1$ independent components. Even if spherical multi-pole moments are preferred in calculations of molecular interactions, spherical harmonics have various disadvantages in numerical calculations: for analytical and numerical calculations Cartesian moments are often more convenient (see for instance Ref.[54] for the case of the electrostatic potential). It is therefore preferable using the irreducible Cartesian STF tensors [55] (having $2l + 1$

independent components if of rank l), which are obtained by using *Cartesian spherical (or solid) harmonic tensors* in place of spherical harmonics.

Given an Euclidean tensor $A_{k_1...k_I}$ on R^3, one defines the completely symmetrized tensor $S_{k_1..k_I} \equiv A_{(k_1..k_I)} = \frac{1}{I!}\sum_\pi A_{k_{\pi(1)}...k_{\pi(I)}}$. Then, the associated STF tensor is obtained by removing all traces ($[I/2]$ = largest integer $\leq I/2$)

$$
\begin{aligned}
A^{(STF)}_{k_1...k_I} &= \sum_{n=0}^{[I/2]} a_n\, \delta_{(k_1k_2}...\delta_{k_{2n-1}k_{2n}} S_{k_{2n+1}...k_I)i_1i_1...j_nj_n},\\
a_n &\equiv (-1)^n \frac{l!(2l-2n-1)!!}{(l-2n)!(2l-1)!!(2n)!!}.
\end{aligned}
\tag{4.34}
$$

For instance $(T_{abc})^{STF} \equiv T_{(abc)} - \frac{1}{5}\Big[\delta_{ab}T_{(iic)} + \delta_{ac}T_{(ibi)} + \delta_{bc}T_{(aii)}\Big]$.

C. Open N-body systems.

Consider now an open sub-system of the isolated system of N charged positive-energy particles plus the electro-magnetic field in the radiation gauge (see the second paper of Ref.[6]). The energy-momentum tensor and the Hamilton equations on the Wigner hyperplane are, respectively, [to avoid degenerations we assume that all the masses m_i are different; $\vec{\pi}_\perp = \vec{E}_\perp$]

$$
\begin{aligned}
T^{\tau\tau}(\tau,\vec{\sigma}) &= \sum_{i=1}^N \delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\sqrt{m_i^2 + [\vec{\kappa}_i(\tau) - Q_i\vec{A}_\perp(\tau,\vec{\eta}_i(\tau))]^2} +\\
&+ \sum_{i=1}^N Q_i\,\vec{\pi}_\perp(\tau,\vec{\sigma}) \times \frac{\vec{\partial}}{\triangle}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau)) + \frac{1}{2}[\vec{\pi}_\perp^2 + \vec{B}^2](\tau,\vec{\sigma}) +\\
&+ \frac{1}{2}\sum_{i,k,i\neq k}^{1..N} Q_i\,Q_k \frac{\vec{\partial}}{\triangle}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau)) \cdot \frac{\vec{\partial}}{\triangle}\delta^3(\vec{\sigma}-\vec{\eta}_k(\tau)),
\end{aligned}
$$

$$
\begin{aligned}
T^{r\tau}(\tau,\vec{\sigma}) &= \sum_{i=1}^N \delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))[\kappa_i^r(\tau) - Q_iA_\perp^r(\tau,\vec{\eta}_i(\tau))] +\\
&+ [\Big(\vec{\pi}_\perp + \sum_{i=1}^N Q_i\frac{\vec{\partial}}{\triangle}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\Big) \times \vec{B}](\tau,\vec{\sigma}),\\
T^{rs}(\tau,\vec{\sigma}) &= \sum_{i=1}^N \delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\frac{[\kappa_i^r(\tau) - Q_iA_\perp^r(\tau,\vec{\eta}_i(\tau))][\kappa_i^s(\tau) - Q_iA_\perp^s(\tau,\vec{\eta}_i(\tau))]}{\sqrt{m_i^2 + [\vec{\kappa}_i(\tau) - Q_i\vec{A}_\perp(\tau,\vec{\eta}_i(\tau))]^2}} -\\
&- \Big[\frac{1}{2}\delta^{rs}[\Big(\vec{\pi}_\perp + \sum_{i=1}^N Q_i\frac{\vec{\partial}}{\triangle}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\Big)^2 + \vec{B}^2] -
\end{aligned}
$$

$$
\begin{aligned}
&- \ [\Big(\vec{\pi}_\perp + \sum_{i=1}^N Q_i \frac{\vec{\partial}}{\triangle}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\Big)^r \Big(\vec{\pi}_\perp + \sum_{i=1}^N Q_i \frac{\vec{\partial}}{\triangle}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\Big)^s + \\
&+ \ B^r B^s]\Big](\tau,\vec{\sigma}).
\end{aligned} \tag{4.35}
$$

$$
\begin{aligned}
\dot{\vec{\eta}}_i(\tau) \ &\stackrel{\circ}{=} \ \frac{\vec{\kappa}_i(\tau) - Q_i \vec{A}_\perp(\tau,\vec{\eta}_i(\tau))}{\sqrt{m_i^2 + (\vec{\kappa}_i(\tau) - Q_i \vec{A}_\perp(\tau,\vec{\eta}_i(\tau)))^2}}, \\
\dot{\vec{\kappa}}_i(\tau) \ &\stackrel{\circ}{=} \ \sum_{k\not= i} \frac{Q_i Q_k (\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau))}{4\pi \mid \vec{\eta}_i(\tau) - \vec{\eta}_k(\tau) \mid^3} + Q_i\, \dot{\eta}_i^u(\tau)\, \frac{\partial}{\partial \vec{\eta}_i} A^u_\perp(\tau,\vec{\eta}_i(\tau))],
\end{aligned}
$$

$$
\dot{A}_{\perp r}(\tau,\vec{\sigma}) \stackrel{\circ}{=} -\pi_{\perp r}(\tau,\vec{\sigma}),
$$

$$
\dot{\pi}^r_\perp(\tau,\vec{\sigma}) \ \stackrel{\circ}{=} \ \Delta A^r_\perp(\tau,\vec{\sigma}) - \sum_i Q_i P^{rs}_\perp(\vec{\sigma}) \dot{\eta}_i^s(\tau) \delta^3(\vec{\sigma}-\vec{\eta}_i(\tau)),
$$

$$
\vec{\kappa}_+(\tau) \ + \ \int d^3\sigma [\vec{\pi}_\perp \times \vec{B}](\tau,\vec{\sigma}) \approx 0 \ (rest - frame\, condition). \tag{4.36}
$$

Let us note that in this reduced phase space there are only either particle-field interactions or action-at-a-distance 2-body interactions. The particle world-lines are $x^\mu_i(\tau) = x^\mu_o + u^\mu(p_s)\, \tau + \epsilon^\mu_r(u(p_s))\, \eta^r_i(\tau)$, while their 4-momenta are $p^\mu_i(\tau) = \sqrt{m_i^2 + [\vec{\kappa}_i - Q_i\, \vec{A}_\perp(\tau,\vec{\eta}_i)]^2}\, u^\mu(p_s) + \epsilon^\mu_r(u(p_s))\, [\kappa^r_i - Q_i\, A^r_\perp(\tau,\vec{\eta}_i)]$.

The generators of the internal Poincaré group are

$$
\begin{aligned}
\mathcal{P}^\tau_{(int)} \ &= \ M = \sum_{i=1}^N \sqrt{m_i^2 + (\vec{\kappa}_i(\tau) - Q_i \vec{A}_\perp(\tau,\vec{\eta}_i(\tau)))^2} + \\
&+ \ \frac{1}{2} \sum_{i\not= j} \frac{Q_i Q_j}{4\pi \mid \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \mid} + \int d^3\sigma \frac{1}{2}[\vec{\pi}^2_\perp + \vec{B}^2](\tau,\vec{\sigma}), \\
\vec{\mathcal{P}}_{(int)} \ &= \ \vec{\kappa}_+(\tau) + \int d^3\sigma [\vec{\pi}_\perp \times \vec{B}](\tau,\vec{\sigma}) \approx 0, \\
\mathcal{J}^r_{(int)} \ &= \ \sum_{i=1}^N (\vec{\eta}_i(\tau) \times \vec{\kappa}_i(\tau))^r + \int d^3\sigma\, (\vec{\sigma} \times\ [\vec{\pi}_\perp \times \vec{B}]^r(\tau,\vec{\sigma}), \\
\mathcal{K}^r_{(int)} \ &= \ -\sum_{i=1}^N \vec{\eta}_i(\tau) \sqrt{m_i^2 + [\vec{\kappa}_i(\tau) - Q_i \vec{A}_\perp(\tau,\ \vec{\eta}_i(\tau))]^2} +
\end{aligned}
$$

$$+\ \frac{1}{2}\Big[Q_i \sum_{i=1}^{N}\sum_{j\neq i}^{1..N} Q_j \int d^3\sigma\, \sigma^r\, \vec{c}(\vec{\sigma}-\vec{\eta}_i(\tau))\cdot \vec{c}(\vec{\sigma}-\vec{\eta}_j(\tau)) +$$

$$+\ Q_i \int d^3\sigma \pi^r_{\perp}(\tau,\vec{\sigma}) c(\vec{\sigma}-\ \vec{\eta}_i(\tau))\Big] - \frac{1}{2}\int d^3\sigma \sigma^r\,(\vec{\pi}^2_{\perp}+\vec{B}^2)(\tau,\vec{\sigma}), \qquad (4.37)$$

with $c(\vec{\eta}_i-\vec{\eta}_j) = 1/(4\pi|\vec{\eta}_j-\vec{\eta}_i|)$ $[\triangle\, c(\vec{\sigma}) = \delta^3(\vec{\sigma})$, $\triangle = -\vec{\partial}^2$, $\vec{c}(\vec{\sigma}) = \vec{\partial}\, c(\vec{\sigma}) = \vec{\sigma}/(4\pi\,|\vec{\sigma}|^3)]$.

$\mathcal{P}^{\tau}_{(int)} = q^{\tau\tau}$ and $\mathcal{P}^{r}_{(int)} = q^{r\tau}$ are the mass and momentum monopoles, respectively.

For the sake of simplicity, consider the sub-system formed by the two particles of mass m_1 and m_2. Our considerations may be extended to any cluster of particles. This sub-system is *open*: besides their mutual interaction the two particles have Coulomb interaction with the other $N-2$ particles and are affected by the transverse electric and magnetic fields.

Exploiting the multi-poles we will select a set of *effective parameters* (mass, 3-center of motion, 3-momentum, spin) describing the two-particle cluster as a global entity subject to external forces in the global rest-frame instant form. This was indeed the original motivation of the multipolar expansion in general relativity: replacing an extended object (an open system due to the presence of the gravitational field) by a set of multi-poles concentrated on a center of motion. Now, in the rest-frame instant form it is possible to show that there is no preferred centroid for open system so that, unlike the case of isolated systems where, in the rest frame $\vec{\kappa}_+ \approx 0$, all possible conventions identify the same centroid, different centers of motion can be selected according to different conventions. We will see, however, that one specific choice exists showing preferable properties.

Given the energy-momentum tensor $T^{AB}(\tau,\vec{\sigma})$ (4.35) of the isolated system, it would seem natural to define *the energy-momentum tensor* $T^{AB}_{c(n)}(\tau,\vec{\sigma})$ *of an open sub-system composed by a cluster of* $n \leq N$ *particles* as the sum of all the terms in Eq.(4.35) containing a dependence on the variables $\vec{\eta}_i$, $\vec{\kappa}_i$, of the particles of the cluster. Besides kinetic terms, this tensor would contain internal mutual interactions as well as external interactions of the cluster particles with the environment composed by the other $N-n$ particles and by the transverse electro-magnetic field. There is an ambiguity, however. While there is no problem in attributing to the cluster the whole interaction with the electro-magnetic field, why should we attribute to it just *all the external interactions with the other* $N-n$ *particles*? Since we have 2-body interactions, it seems more reasonable to attribute only *half* of these external interactions to the cluster and consider the other half as a property of the remaining $N-n$ particles. Let us remark that considering, e.g., two clusters composed by two non-overlapping sets of n_1 and n_2 particles, respectively, since the mutual Coulomb interactions between the clusters are present in both $T^{AB}_{c(n_1)}$ and $T^{AB}_{c(n_2)}$, according to the first choice we would get $T^{AB}_{c(n_1+n_2)} \neq T^{AB}_{c(n_1)} + T^{AB}_{c(n_2)}$. On the other hand, according to the second choice we get $T^{AB}_{c(n_1+n_2)} = T^{AB}_{c(n_1)} + T^{AB}_{c(n_2)}$. Since this property is important for studying the mutual relative motion of two clusters in actual cases, we will adopt *the convention that the energy-momentum tensor of a n particle cluster contains only half of the external interaction with the other* $N-n$ *particles*.

Let us remark that, in the case of k-body forces, this convention should be replaced by the following rule: i) for each particle m_i of the cluster and each k-body term in the energy-momentum tensor involving this particle, $k = h_i + (k - h_i)$, where h_i is the number of particles of the cluster participating to this particular k-body interaction; ii) only the fraction h_i/k of this particular k-body interaction term containing m_i must be attributed to the cluster.

Let us consider the cluster composed by the two particles with mass m_1 and m_2. The knowledge of $T_c^{AB} \stackrel{def}{=} T_{c(2)}^{AB}$ on the Wigner hyper-plane of the global rest-frame instant form allows us to find the following 10 *non conserved* charges [due to $Q_i^2 = 0$ we have $\sqrt{m_i^2 + [\vec{\kappa}_i - Q_i\, \vec{A}_{\perp}(\tau, \vec{\eta}_i)]^2} = \sqrt{m_i^2 + \vec{\kappa}_1^2} - Q_i\, \frac{\vec{\kappa}_i \cdot \vec{A}_{\perp}(\tau, \vec{\eta}_i)}{\sqrt{m_i^2 + \vec{\kappa}_i^2}}$]

$$
\begin{aligned}
M_c &= \int d^3\sigma\, T_c^{\tau\tau}(\tau, \vec{\sigma}) = \sum_{i=1}^2 \sqrt{m_i^2 + [\vec{\kappa}_i(\tau) - Q_i\, \vec{A}_{\perp}(\tau, \vec{\eta}_i(\tau))]^2} + \\
&+ \frac{Q_1\, Q_2}{4\pi\, |\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau)|^2} + \frac{1}{2} \sum_{i=1}^2 \sum_{k \neq 1,2} \frac{Q_i\, Q_k}{4\pi\, |\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)|^2} = \\
&= M_{c(int)} + M_{c(ext)}, \\
&\quad M_{c(int)} = \sum_{i=1}^2 \sqrt{m_i^2 + \vec{\kappa}_i^2} - \frac{Q_1\, Q_2}{4\pi\, |\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau)|^2}, \\
\vec{\mathcal{P}}_c &= \{\int d^3\sigma\, T_c^{r\tau}(\tau, \vec{\sigma})\} = \vec{\kappa}_1(\tau) + \vec{\kappa}_2(\tau), \\
\vec{\mathcal{J}}_c &= \{\epsilon^{ruv} \int d^3\sigma\, [\sigma^u\, T_c^{v\tau} - \sigma^v\, T_c^{u\tau}](\tau, \vec{\sigma})\} = \vec{\eta}_i(\tau) \times \vec{\kappa}_1(\tau) + \vec{\eta}_2(\tau) \times \vec{\kappa}_2(\tau), \\
\vec{\mathcal{K}}_c &= -\int d^3\sigma\, \vec{\sigma}\, T_c^{\tau\tau}(\tau, \vec{\sigma}) = -\sum_{i=1}^2 \vec{\eta}_i(\tau) \sqrt{m_i^2 + [\vec{\kappa}_i(\tau) - Q_i\, \vec{A}_{\perp}(\tau, \vec{\eta}_i(\tau))]^2} - \\
&- \sum_{i=1}^2 Q_i \int d^3\sigma\, \vec{\pi}_{\perp}(\tau, \vec{\sigma})\, c(\vec{\sigma} - \vec{\eta}_i(\tau)) - \\
&- Q_1\, Q_2 \int d^3\sigma\, \vec{\sigma}\, \vec{c}(\vec{\sigma} - \vec{\eta}_1(\tau)) \cdot \vec{c}(\vec{\sigma} - \vec{\eta}_2(\tau)) - \\
&- \frac{1}{2} \sum_{i=1}^2 Q_i \sum_{k \neq 1,2} Q_k \int d^3\sigma\, \vec{\sigma}\, \vec{c}(\vec{\sigma} - \vec{\eta}_i(\tau)) \cdot \vec{c}(\vec{\sigma} - \vec{\eta}_k(\tau)) = \\
&= \vec{\mathcal{K}}_{c(int)} + \vec{\mathcal{K}}_{c(ext)}, \\
&\quad \vec{\mathcal{K}}_{c(int)} = -\sum_{i=1}^2 \vec{\eta}_i(\tau) \sqrt{m_i^2 + \vec{\kappa}_i^2} - Q_1\, Q_2 \int d^3\sigma\, \vec{\sigma}\, \vec{c}(\vec{\sigma} - \vec{\eta}_1(\tau)) \cdot \vec{c}(\vec{\sigma} - \vec{\eta}_2(\tau)).
\end{aligned}
\tag{4.38}
$$

Such charges do not satisfy the algebra of an internal Poincare' group just because of the openness of the system. Working in an instant form of dynamics, only the cluster internal energy and boosts depend on the (internal and external) interactions. Again, $M_c = q_c^{\tau\tau}$ and $\mathcal{P}_c^r = q_c^{r\tau}$ are the mass and momentum monopoles of the cluster.

Another quantity to be considered is the momentum dipole

$$
\begin{aligned}
p_c^{ru} &= \int d^3\sigma\, \sigma^r\, T_c^{u\tau}(\tau, \vec{\sigma}) = \\
&= \sum_{i=1}^{2} \eta_i^r(\tau)\, \kappa_i^u(\tau) - \sum_{i=1}^{2} Q_i \int d^3\sigma\, c(\vec{\sigma} - \vec{\eta}_i(\tau))\, [\partial^r\, A_\perp^s + \partial^s\, A_\perp^r](\tau, \vec{\sigma}), \\
&\quad p_c^{ru} + p_c^{ur} = \sum_{i=1}^{2} [\eta_i^r(\tau)\, \kappa_i^u(\tau) + \eta_i^u(\tau)\, \kappa_i^r(\tau)] - \\
&- 2 \sum_{i=1}^{2} Q_i \int d^3\sigma\, c(\vec{\sigma} - \vec{\eta}_i(\tau))\, [\partial^r\, A_\perp^s + \partial^s\, A_\perp^r](\tau, \vec{\sigma}), \\
&\quad p_c^{ru} - p_c^{ur} = \epsilon^{ruv}\, \mathcal{J}_c^v.
\end{aligned}
\tag{4.39}
$$

The time variation of the 10 charges (4.38) can be evaluated by using the equations of motion (4.36)

$$
\begin{aligned}
\frac{dM_c}{d\tau} &= \sum_{i=1}^{2} Q_i \Big(\frac{\vec{\kappa}_i(\tau) \cdot \vec{\pi}_\perp(\tau, \vec{\eta}_i(\tau))}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} + \\
&+ \frac{1}{2} \sum_{k \neq 1,2} Q_k \Big[\frac{\vec{\kappa}_i(\tau)}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} + \frac{\vec{\kappa}_k(\tau)}{\sqrt{m_k^2 + \vec{\kappa}_k^2}} \Big] \cdot \frac{\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)}{4\pi\, |\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)|^3} \Big), \\
\frac{d\mathcal{P}_c^r}{d\tau} &= \sum_{i=1}^{2} Q_i \Big(\frac{\vec{\kappa}_i(\tau)}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \frac{\partial A_\perp^r(\tau, \vec{\eta}_i(\tau))}{\partial \vec{\eta}_i} + \sum_{k \neq 1,2} Q_k\, \frac{\eta_i^r(\tau) - \eta_k^r(\tau)}{4\pi\, |\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)|^3} \Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\vec{\mathcal{J}}_c}{d\tau} &= \sum_{i=1}^{2} Q_i \Big(\frac{\vec{\kappa}_i(\tau)}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \times \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) + \vec{\eta}_i(\tau) \times \Big[\frac{\vec{\kappa}_i(\tau)}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \frac{\partial}{\partial \vec{\eta}_i} \Big] \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) - \\
&- \sum_{k \neq i} Q_k\, \frac{\vec{\eta}_i(\tau) \times \vec{\eta}_k(\tau)}{4\pi\, |\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)|^3} \Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\mathcal{K}_c^r}{d\tau} &= Q_1 Q_2 \int d^3\sigma\, \vec{\sigma}\, \Big(\Big[\Big(\frac{\vec{\kappa}_1(\tau)}{\sqrt{m_1^2 + \vec{\kappa}_1^2(\tau)}} \cdot \vec{\partial} \Big)\, \vec{c}(\vec{\sigma} - \vec{\eta}_1(\tau)) \Big] \cdot \vec{c}(\vec{\sigma} - \vec{\eta}_2(\tau)) + \\
&+ \vec{c}(\vec{\sigma} - \vec{\eta}_1(\tau)) \cdot \Big[\Big(\frac{\vec{\kappa}_2(\tau)}{\sqrt{m_2^2 + \vec{\kappa}_2^2(\tau)}} \cdot \vec{\partial} \Big)\, \vec{c}(\vec{\sigma} - \vec{\eta}_2(\tau)) \Big] \Big) -
\end{aligned}
$$

$$
- \frac{1}{2} \sum_{i=1}^{2} Q_i \sum_{k\neq 1,2} Q_k \int d^3\sigma\, \vec{\sigma} \Big(\Big[\Big(\frac{\vec{\kappa}_i(\tau)}{\sqrt{m_i^2 + \vec{\kappa}_i^2(\tau)}} \cdot \vec{\partial} \Big)\, \vec{c}(\vec{\sigma} - \vec{\eta}_i(\tau)) \Big] \cdot \vec{c}(\vec{\sigma} - \vec{\eta}_k(\tau)) +
$$

$$
+\ \vec{c}(\vec{\sigma} - \vec{\eta}_i(\tau)) \cdot \Big[\Big(\frac{\vec{\kappa}_k(\tau)}{\sqrt{m_k^2 + \vec{\kappa}_k^2(\tau)}} \cdot \vec{\partial} \Big)\, \vec{c}(\vec{\sigma} - \vec{\eta}_k(\tau)) \Big] \Big). \tag{4.40}
$$

Note that, if we have two clusters of n_1 and n_2 particles respectively, our definition of cluster energy-momentum tensor implies

$$
\begin{aligned}
M_{c(n_1+n_2)} &= M_{c(n_1)} + M_{c(n_2)},\\
\vec{\mathcal{P}}_{c(n_1+n_2)} &= \vec{\mathcal{P}}_{c(n_1)} + \vec{\mathcal{P}}_{c(n_2)},\\
\vec{\mathcal{J}}_{c(n_1+n_2)} &= \vec{\mathcal{J}}_{c(n_1)} + \vec{\mathcal{J}}_{c(n_2)},\\
\vec{\mathcal{K}}_{c(n_1+n_2)} &= \vec{\mathcal{K}}_{c(n_1)} + \vec{\mathcal{K}}_{c(n_2)}.
\end{aligned} \tag{4.41}
$$

The main problem is now the determination of an *effective center of motion* $\zeta^r_c(\tau)$ with world-line $w^\mu_c(\tau) = x^\mu_o + u^\mu(p_s)\,\tau + \epsilon^\mu_r(u(p_s))\,\zeta^r_c(\tau)$ in the gauge $T_s \equiv \tau$, $\vec{q}_+ = \vec{R}_+ = \vec{y}_+ \equiv 0$ of the isolated system. The unit 4-velocity of this center of motion is $u^\mu_c(\tau) = \dot{w}^\mu_c(\tau)/\sqrt{1 - \dot{\vec{\zeta}}^2_c(\tau)}$ with $\dot{w}^\mu_c(\tau) = u^\mu(p_s) + \epsilon^\mu_r(u(p_s))\,\dot{\zeta}^r_c(\tau)$. By using $\delta\, z^\mu(\tau, \vec{\sigma}) = \epsilon^\mu_r(u(p_s))\,(\sigma^r - \zeta^r(\tau))$ we can define the multipoles of the cluster with respect to the world-line $w^\mu_c(\tau)$

$$
q_c^{r_1..r_n AB}(\tau) = \int d^3\sigma\, [\sigma^{r_1} - \zeta_c^{r_1}(\tau)]..[\sigma^{r_n} - \zeta_c^{r_n}(\tau)]\, T_c^{AB}(\tau, \vec{\sigma}). \tag{4.42}
$$

The mass and momentum monopoles, and the mass, momentum and spin dipoles are, respectively

$$
\begin{aligned}
q_c^{\tau\tau} &= M_c, \qquad q_c^{r\tau} = \mathcal{P}^r_c,\\
q_c^{r\tau\tau} &= -\mathcal{K}^r_c - M_c\,\zeta^r_c(\tau) = M_c\,(R^r_c(\tau) - \zeta^r_c(\tau)), \qquad q_c^{ru\tau} = p_c^{ru}(\tau) - \zeta^r_c(\tau)\,\mathcal{P}^u_c,
\end{aligned}
$$

$$
\begin{aligned}
S_c^{\mu\nu} &= [\epsilon^\mu_r(u(p_s))\, u^\nu(p_s) - \epsilon^\nu_r(u(p_s))\, u^\mu(p_s)]\, q_c^{r\tau\tau} + \epsilon^\mu_r(u(p_s))\, \epsilon^\nu_u(u(p_s))\, (q_c^{ru\tau} - q_c^{ur\tau}) =\\
&= [\epsilon^\mu_r(u(p_s))\, u^\nu(p_s) - \epsilon^\nu_r(u(p_s))\, u^\mu(p_s)]\, M_c\,(R^r_c - \zeta^r_c) +\\
&+\ \epsilon^\mu_r(u(p_s))\, \epsilon^\nu_u(u(p_s))\, \Big[\epsilon^{ruv}\, \mathcal{J}^v_c - (\zeta^r_c\, \mathcal{P}^u_c - \zeta^u_c\, \mathcal{P}^r_c) \Big],\\
&\Rightarrow\ m^\mu_{c(p_s)} = -S_c^{\mu\nu}\, u_\nu(p_s) = -\epsilon^\mu_r(u(p_s))\, q_c^{r\tau\tau}.
\end{aligned} \tag{4.43}
$$

Then, consider the following possible definitions of effective centers of motion (clearly, many other possibilities exist)

1) *Center of energy as center of motion*, $\vec{\zeta}_{c(E)}(\tau) = \vec{R}_c(\tau)$, where $\vec{R}_c(\tau)$ is *a 3-center of energy* for the cluster, built by means of the standard definition

$$\vec{R}_c = -\frac{\vec{\mathcal{K}}_c}{M_c}. \tag{4.44}$$

It is determined by the requirement that either the mass dipole vanishes, $q_c^{r\tau\tau} = 0$, or the mass dipole moment with respect to $u^\mu(p_s)$ vanishes, $m^\mu_{c(p_s)} = 0$.

The center of energy seems to be the only center of motion enjoying the simple composition rule

$$\vec{R}_{c(n_1+n_2)} = \frac{M_{c(n_1)}\,\vec{R}_{c(n_1)} + M_{c(n_2)}\,\vec{R}_{c(n_2)}}{M_{c(n_1+n_2)}}. \tag{4.45}$$

The constitutive relation between $\vec{\mathcal{P}}_c$ and $\dot{\vec{R}}_c(\tau)$, see Eq.(4.23), is

$$\begin{aligned}
0 &= \frac{dq_c^{r\tau\tau}}{d\tau} = -\dot{\mathcal{K}}^r_c - \dot{M}_c\,R^r_c - M_c\,\dot{R}^r_c,\\
&\Downarrow\\
\vec{\mathcal{P}}_c &= Q_1Q_2\int d^3\sigma\,\vec{\sigma}\,\Big(\Big[\Big(\frac{\vec{\kappa}_1(\tau)}{\sqrt{m_1^2+\vec{\kappa}_1^2(\tau)}}\cdot\vec{\partial}\Big)\,\vec{c}(\vec{\sigma}-\vec{\eta}_1(\tau))\Big]\cdot\vec{c}(\vec{\sigma}-\vec{\eta}_2(\tau)) +\\
&+ \vec{c}(\vec{\sigma}-\vec{\eta}_1(\tau))\cdot\Big[\Big(\frac{\vec{\kappa}_2(\tau)}{\sqrt{m_2^2+\vec{\kappa}_2^2(\tau)}}\cdot\vec{\partial}\Big)\,\vec{c}(\vec{\sigma}-\vec{\eta}_2(\tau))\Big]\Big) -\\
&- \frac{1}{2}\sum_{i=1}^2 Q_i\sum_{k\not=1,2} Q_k\int d^3\sigma\,\vec{\sigma}\,\Big(\Big[\Big(\frac{\vec{\kappa}_i(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}}\cdot\vec{\partial}\Big)\,\vec{c}(\vec{\sigma}-\vec{\eta}_i(\tau))\Big]\cdot\vec{c}(\vec{\sigma}-\vec{\eta}_k(\tau)) +\\
&+ \vec{c}(\vec{\sigma}-\vec{\eta}_i(\tau))\cdot\Big[\Big(\frac{\vec{\kappa}_k(\tau)}{\sqrt{m_k^2+\vec{\kappa}_k^2(\tau)}}\cdot\vec{\partial}\Big)\,\vec{c}(\vec{\sigma}-\vec{\eta}_k(\tau))\Big]\Big).
\end{aligned} \tag{4.46}$$

¿From Eq.(4.25), it follows that the associated cluster spin tensor is

$$\begin{aligned}
S^{\mu\nu}_c &= \epsilon^\mu_r(u(p_s))\,\epsilon^\nu_u(u(p_s))\,[q_c^{ru\tau} - q_c^{ur\tau}] =\\
&= \epsilon^\mu_r(u(p_s))\,\epsilon^\nu_u(u(p_s))\,\epsilon^{ruv}\Big[\mathcal{J}^v_c - (\vec{R}_c\times\vec{\mathcal{P}}_c)^v\Big].
\end{aligned} \tag{4.47}$$

2) *Pirani centroid $\vec{\zeta}_{c(P)}(\tau)$ as center of motion.* It is determined by the requirement that the mass dipole moment with respect to 4-velocity $\dot{w}^\mu_c(\tau)$ vanishes (it involves the anti-symmetric part of p^{ur}_c)

$$
\begin{aligned}
m^\mu_{c(\dot w_c)} &= -S^{\mu\nu}_c\, \dot w_{c\nu} = 0, \quad \Rightarrow \dot{\vec \zeta}_{c(P)} \cdot \vec \zeta_{c(P)} = \dot{\vec \zeta}_{c(P)} \cdot \vec R_c,\\
&\Downarrow\\
\vec \zeta_{c(P)}(\tau) &= \frac{1}{M_c - \vec{\mathcal{P}}_c \cdot \dot{\vec \zeta}_{c(P)}(\tau)} \Big[M_c\, \vec R_c - \vec R_c \cdot \dot{\vec \zeta}_{c(P)}(\tau)\, \vec{\mathcal{P}}_c - \dot{\vec \zeta}_{c(P)}(\tau) \times \vec{\mathcal{J}}_c \Big]. \qquad (4.48)
\end{aligned}
$$

Therefore this centroid is implicitly defined as the solution of these three coupled first order ordinary differential equations.

3) *Tulczyjew centroid $\vec \zeta_{c(T)}(\tau)$ as center of motion.* If we define the cluster 4-momentum $P^\mu_c = M_c\, u^\mu(p_s) + \mathcal{P}^s_c\, \epsilon^\mu_s(u(p_s))$ $[P^2_c = M^2_c - \vec{\mathcal{P}}^2_c \overset{def}{=} \mathcal{M}^2_c]$, its definition is the requirement that the mass dipole moment with respect to P^μ_c vanishes (it involves the anti-symmetric part of p^{ur}_c)

$$
\begin{aligned}
m^\mu_{c(P_c)} &= -S^{\mu\nu}_c\, P_{c\nu} = 0, \quad \Rightarrow \vec{\mathcal{P}}_c \cdot \vec \zeta_{c(T)} = \vec{\mathcal{P}}_c \cdot \vec R_c,\\
&\Downarrow\\
\vec \zeta_{c(T)}(\tau) &= \frac{1}{M^2_c - \vec{\mathcal{P}}^2_c} \Big[M^2_c\, \vec R_c - \vec{\mathcal{P}}_c \cdot \vec R_c\, \vec{\mathcal{P}}_c - \vec{\mathcal{P}}_c \times \vec{\mathcal{J}}_c \Big]. \qquad (4.49)
\end{aligned}
$$

Let us show that this centroid satisfies the free particle relation as constitutive relation

$$
\begin{aligned}
\vec{\mathcal{P}}_c &= M_c\, \dot{\vec \zeta}_{c(T)}, \quad \Rightarrow \quad P^\mu_c = M_c \Big[u^\mu(p_s) + \dot\zeta^s_{c(T)}\, \epsilon^\mu_s(u(p_s)) \Big],\\
&\quad q^{r\tau\tau}_{c(T)} = \frac{M_c}{M^2_c - \vec{\mathcal{P}}^2_c} \Big[\vec{\mathcal{P}}^2_c\, \vec R_c + \vec{\mathcal{P}}_c \cdot \vec R_c\, \vec{\mathcal{P}}_c + \vec{\mathcal{P}}_c \times \vec{\mathcal{J}}_c \Big],\\
&\quad S^{\mu\nu}_c = [\epsilon^\mu_r(u(p_s))\, u^\nu(p_s) - \epsilon^\nu_r(u(p_s))\, u^\mu(p_s)]\, q^{r\tau\tau}_{c(T)} +\\
&\qquad + \epsilon^\mu_r(u(p_s))\, \epsilon^\nu_u(u(p_s))\, \epsilon^{ruv} \Big[\mathcal{J}^v_c - (\vec \zeta_{c(T)} \times \vec{\mathcal{P}}_c)^v \Big]. \qquad (4.50)
\end{aligned}
$$

If we use Eq.(4.48) to find a Pirani centroid such that $\dot{\vec \zeta}_c = \vec{\mathcal{P}}_c / M_c$, it turns out that the condition (4.48) becomes Eq.(4.49) and this implies Eq.(4.50).

The equations of motion

$$
M_c(\tau)\, \ddot{\vec \zeta}_{c(T)}(\tau) = \dot{\vec{\mathcal{P}}}_c(\tau) - \dot M_c(\tau)\, \dot{\vec \zeta}_{c(T)}(\tau), \qquad (4.51)
$$

contain both internal and external forces. In spite of the nice properties (4.50) and (4.51) of the Tulczyjew centroid, this effective center of motion fails to satisfy a simple composition property. The relation among the Tulczyjew centroids of clusters with n_1, n_2 and $n_1 + n_2$

particles respectively is much more complicated of the composition (4.45) of the centers of energy.

All the previous centroids coincide for an isolated system in the rest-frame instant form with $\vec{\mathcal{P}}_c = \vec{\kappa}_+ \approx 0$ in the gauge $\vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+ \approx 0$.

4) The *Corinaldesi-Papapetrou centroid with respect to a time-like observer with 4-velocity* $v^\mu(\tau)$, $\vec{\zeta}^{(v)}_{c(CP)}(\tau)$ *as center of motion.*

$$m^\mu_{c(v)} = -S^{\mu\nu}_c \, v_\nu = 0. \tag{4.52}$$

Clearly these centroids are unrelated to the previous ones being dependent on the choice of an arbitrary observer.

5) The *Pryce center of spin or classical canonical Newton-Wigner centroid* $\vec{\zeta}_{c(NW)}$. It defined as the solution of the differential equations implied by the requirement $\{\zeta^r_{c(NW)}, \zeta^s_{c(NW)}\} = 0$, $\{\zeta^r_{c(NW)}, \mathcal{P}^s_c\} = \delta^{rs}$. Let us remark that, being in an instant form of dynamics, we have $\{\mathcal{P}^r_c, \mathcal{P}^s_c\} = 0$ also for an open system.

The two effective centers of motion which appear to be more useful for applications are the center of energy $\vec{\zeta}_{c(E)}(\tau)$ and Tulczyjew's centroid $\vec{\zeta}_{c(T)}(\tau)$, with $\vec{\zeta}_{c(E)}(\tau)$ preferred for the study of the mutual motion of clusters due to Eq.(4.45).

Therefore, in the spirit of the multipolar expansion, our two-body cluster may be described by an effective non-conserved internal energy (or mass) $M_c(\tau)$, by the world-line $w^\mu_c = x^\mu_o + u^\mu(p_s)\,\tau + \epsilon^\mu_r(u(p_s))\,\zeta^r_{c(E\,or\,T)}(\tau)$ associated with the effective center of motion $\vec{\zeta}_{c(E\,or\,T)}(\tau)$ and by the effective 3-momentum $\vec{\mathcal{P}}_c(\tau)$, with $\vec{\zeta}_{c(E\,or\,T)}(\tau)$ and $\vec{\mathcal{P}}_c(\tau)$ forming a non-canonical basis for the collective variables of the cluster. A non-canonical effective spin for the cluster in the 1) and 3) cases is defined by

a) case of the center of energy:

$$\begin{aligned}
\vec{\mathcal{S}}_{c(E)}(\tau) &= \vec{\mathcal{J}}_c(\tau) - \vec{R}_c(\tau) \times \vec{\mathcal{P}}_c(\tau),\\
\frac{d\vec{\zeta}_{c(E)}(\tau)}{d\tau} &= \frac{d\vec{\mathcal{J}}_c(\tau)}{d\tau} - \frac{d\vec{R}_c(\tau)}{d\tau} \times \vec{\mathcal{P}}_c(\tau) - \vec{R}_c(\tau) \times \frac{d\vec{\mathcal{P}}_c(\tau)}{d\tau},
\end{aligned}$$

b) case of the Tulczyjew centroid:

$$\begin{aligned}
\vec{\mathcal{S}}_{c(T)}(\tau) &= \vec{\mathcal{J}}_c(\tau) - \vec{\zeta}_{c(T)}(\tau) \times \vec{\mathcal{P}}_c(\tau) =\\
&= \frac{M^2_c(\tau)\,\vec{\mathcal{S}}_{c(E)}(\tau) - \vec{\mathcal{P}}_c(\tau)\cdot\vec{\mathcal{J}}_c(\tau)\,\vec{\mathcal{P}}_c(\tau)}{M^2_c(\tau) - \vec{\mathcal{P}}^2_c(\tau)},\\
\frac{d\vec{\zeta}_{c(T)}(\tau)}{d\tau} &= \frac{d\vec{\mathcal{J}}_c(\tau)}{d\tau} - \vec{\zeta}_{c(T)}(\tau) \times \frac{d\vec{\mathcal{P}}_c(\tau)}{d\tau}.
\end{aligned} \tag{4.53}$$

Since our cluster contains only two particles, this *pole-dipole description* concentrated on the world-line $w^\mu_c(\tau)$ is equivalent to the original description in terms of the canonical variables $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$ (all the higher multipoles are not independent quantities in this case).

Finally, in Ref.[12] there is an attempt to replace the description of the two body system as an *effective pole-dipole system* with a description as an *effective extended two-body system* by introducing two non-canonical relative variables $\vec{\rho}_{c(E\, or\, T)}(\tau)$, $\vec{\pi}_{c(E\, or\, T)}(\tau)$ with the following definitions

$$\begin{aligned}
\vec{\eta}_1 &\stackrel{def}{=} \vec{\zeta}_{c(E\, or\, T)} + \frac{1}{2}\,\vec{\rho}_{c(E\, or\, T)}, \qquad \vec{\zeta}_{c(E\, or\, T)} = \frac{1}{2}\,(\vec{\eta}_1 + \vec{\eta}_2),\\
\vec{\eta}_2 &\stackrel{def}{=} \vec{\zeta}_{c(E\, or\, T)} - \frac{1}{2}\,\vec{\rho}_{c(E\, or\, T)}, \qquad \vec{\rho}_{c(E\, or\, T)} = \vec{\eta}_1 - \vec{\eta}_2,\\
\vec{\kappa}_1 &\stackrel{def}{=} \frac{1}{2}\,\vec{\mathcal{P}}_c + \vec{\pi}_{c(E\, or\, T)}, \qquad \vec{\mathcal{P}}_c = \vec{\kappa}_1 + \vec{\kappa}_2,\\
\vec{\kappa}_2 &\stackrel{def}{=} \frac{1}{2}\,\vec{\mathcal{P}}_c - \vec{\pi}_{c(E\, or\, T)}, \qquad \vec{\pi}_{c(E\, or\, T)} = \frac{1}{2}\,(\vec{\kappa}_1 - \vec{\kappa}_2),\\
\vec{\mathcal{J}}_c &= \vec{\eta}_1 \times \vec{\kappa}_1 + \vec{\eta}_2 \times \vec{\kappa}_2 = \vec{\zeta}_{c(E\, or\, T)} \times \vec{\mathcal{P}}_c + \vec{\rho}_{c(E\, or\, T)} \times \vec{\pi}_{c(E\, or\, T)},\\
&\Rightarrow\ \vec{\mathcal{S}}_{c(E\, or\, T)} = \vec{\rho}_{c(E\, or\, T)} \times \vec{\pi}_{c(E\, or\, T)}. \qquad (4.54)
\end{aligned}$$

Even if suggested by a canonical transformation, it is *not* a canonical transformation and it only exists because we are working in an instant form of dynamics in which both $\vec{\mathcal{P}}_c$ and $\vec{\mathcal{J}}_c$ do not depend on the interactions. Note that we know everything about this new basis except for the unit vector $\vec{\rho}_{c(E\, or\, T)}/|\vec{\rho}_{c(E\, or\, T)}|$ and the momentum $\vec{\pi}_{c(E\, or\, T)}$. The relevant lacking information can be extracted from the symmetrized momentum dipole $p^{ru}_c + p^{ur}_c$, which is a known effective quantity due to Eq.(4.40). However, strictly speaking, this type of attempt fails, because $p^{ru}_c + p^{ur}_c$ *does not depend only* on the cluster properties but *also* on the external electro-magnetic transverse vector potential at the particle positions, as shown by Eq.(4.39). Consequently, the spin frame, or equivalently the 3 Euler angles associated with the internal spin, depend upon the external fields.

D. The Multipoles of the Real Klein-Gordon Field.

In the rest-frame instant form we have the following expression [26] for the energy-momentum of the real Klein-Gordon field

$$\begin{aligned}
T^{\mu\nu}[x^\mu_s(T_s) + \epsilon^\mu_u(u(p_s))\sigma^u][\phi] &= T^{\mu\nu}[x^\mu_s(T_s) + \epsilon^\mu_u(u(p_s))\sigma^u][X^A_\phi, P^A_\phi, \mathbf{H}, \mathbf{K}] =\\
&= \frac{1}{2}u^\mu(p_s)u^\nu(p_s)[\pi^2 + (\vec{\partial}\phi)^2 + m^2\phi^2](T_s, \vec{\sigma}) +\\
&+ \epsilon^\mu_r(u(p_s))\epsilon^\nu_s(u(p_s))[-\frac{1}{2}\delta_{rs}[\pi^2 - (\vec{\partial}\phi)^2 - m^2\phi^2] +
\end{aligned}$$

$$
\begin{aligned}
&+ \; \partial_r\phi\partial_s\phi](T_s,\vec{\sigma}) - \\
&- \; [u^\mu(p_s)\epsilon^\nu_r(u(p_s)) + u^\nu(p_s)\epsilon^\mu_r(u(p_s))][\pi\partial_r\phi](T_s,\vec{\sigma}) = \\
&= \; \Big[\rho[\phi,\pi]u^\mu(p_s)u^\nu(p_s) + \mathcal{P}[\phi,\pi][\eta^{\mu\nu} - u^\mu(p_s)u^\nu(p_s)] + \\
&+ \; u^\mu(p_s)q^\nu[\phi,\pi] + u^\nu(p_s)q^\mu[\phi,\pi] + \\
&+ \; T^{rs}_{an\, stress}[\phi,\pi]\epsilon^\mu_r(u(p_s))\epsilon^\nu_s(u(p_s))\Big](T_s,\vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
\rho[\phi,\pi] &= \frac{1}{2}[\pi^2 + (\vec{\partial}\phi)^2 + m^2\phi^2], \\
\mathcal{P}[\phi,\pi] &= \frac{1}{2}[\pi^2 - \frac{5}{3}(\vec{\partial}\phi)^2 - m^2\phi^2], \\
q^\mu[\phi,\pi] &= -\pi\partial_r\phi\epsilon^\mu_r(u(p_s)), \\
T^{rs}_{an\, stress}[\phi,\pi] &= -[\partial^r\phi\partial^s\phi - \frac{1}{3}\delta^{rs}(\vec{\partial}\phi)^2], \\
& \delta_{uv}T^{uv}_{an\, stress}[\phi,\pi] = 0,
\end{aligned}
$$

$$
\begin{aligned}
T^{rs}_{stress}(T_s,\vec{\sigma})[\phi] &= \epsilon^r_\mu(u(p_s))\epsilon^s_\nu(u(p_s))T^{\mu\nu}[x^\mu_s(T_s) + \epsilon^\mu_u(u(p_s))\sigma^u][\phi] = \\
&= [\partial^r\phi\partial^s\phi](T_s,\vec{\sigma}) - \frac{1}{2}\delta^{rs}[\pi^2 - (\vec{\partial}\phi)^2 - m^2\phi^2](T_s,\vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
P^\mu_T[\phi] &= \int d^3\sigma T^{\mu\nu}[x^\mu_s(T_s) + \epsilon^\mu_u(u(p_s))\sigma^u][\phi]u_\nu(p_s) = \\
&= P^\tau_\phi u^\mu(p_s) + P^r_\phi\epsilon^\mu_r(u(p_s)) \approx P^\tau_\phi u^\mu(p_s) \approx p^\mu_s, \\
M_\phi &= P^\mu_T[\phi]u_\mu(p_s) = P^\tau_\phi. \qquad (4.55)
\end{aligned}
$$

The stress tensor $T^{rs}_{stress}(T_s,\vec{\sigma})[\phi]$ of the Klein-Gordon field on the Wigner hyper-planes acquires a form reminiscent of the energy-momentum tensor of an ideal relativistic fluid as seen from a local observer at rest (Eckart decomposition; see Ref.[27]): i) the constant normal $u^\mu(p_s)$ to the Wigner hyper-planes replaces the hydrodynamic velocity field of the fluid; ii) $\rho[\phi,\pi](T_s,\vec{\sigma}]$ is the energy density; iii) $\mathcal{P}[\phi,\pi](T_s,\vec{\sigma})$ is the analogue of the pressure (sum of the thermodynamical pressure and of the non-equilibrium bulk stress or viscous pressure); iv) $q^\mu[\phi,\pi](T_s,\vec{\sigma})$ is the analogue of the heat flow; v) $T^{rs}_{an\, stress}[\phi,\pi](T_s,\vec{\sigma})$ is the shear (or anisotropic) stress tensor.

Given the Hamiltonian version of the energy-momentum tensor, Eqs.(4.18) allow us to find all Dixon multi-poles of the Klein-Gordon field [26], with respect to a natural center of motion identified by the collective variable defined in Subsection IIIH. Finally, Eqs.(3.70), conjoined with the assumption that we can interchange the sums with the integrals, allow us to define another class of multi-poles [26] with respect to the centroid $X^\mu_s(\tau)$, origin of the 3-coordinates $\vec{\sigma}$, in the gauge $\vec{X}_\phi \approx 0$, which, when the fields have a compact support in momentum space, form a closed algebra with a generalized Kronecker symbol, that could be quantized instead of the Fourier coefficients.

V. FINAL COMMENTS AND OPEN PROBLEMS.

In this article we have shown how the traditional Jacobi technique based on the clustering of centers of mass for an N-body system, can be profitably replaced by a technique based on the *clustering of spins* in order to develop a theory of the relativistic, rotational, and multi-pole, kinematics for deformable systems. More generally, the relativistic extension is also made possible by a systematic use of the so-called *rest-frame instant form* of relativistic dynamics on Wigner hyper-planes, orthogonal to the conserved 4-momentum of the isolated system. This is a form of dynamics characterized by a typical *doubling* of the Poincaré canonical realizations into so-called *external* and *internal* realizations. In fact, this framework appears to be the most natural theoretical background for the description of isolated systems (particles, strings, fields, fluids) in special relativity.

The rest-frame instant form of the N-body problem has Newton mechanics in the center-of-mass frame as non-relativistic limit. At the same time it is the special relativistic limit of the rest-frame instant form of canonical metric and tetrad gravity [7], when the Newton constant G is turned off. Both in relativistic and non-relativistic theories there is a SO(3) left action on the $(6N-6)$-dimensional phase space of the canonical *relative variables* with respect to the 3-center of mass, generated by the non-Abelian Noether constants for the angular momentum. Correspondingly, the notion of *clustering of the spins of sub-clusters* do exist at both levels, while the Jacobi clustering of the sub-cluster 3-centers of mass is not extendible to special relativity. Previously introduced concepts, like *dynamical body frames* (replacing the standard notion of *body frames* valid for rigid systems), *spin frames*, and *canonical spin bases*, proper of the Galilean group-theoretical and Hamiltonian description of N-body or deformable systems, are likewise directly extended to special relativity. The canonical spin frames and a finite number of dynamical body frames (SO(3) *right actions* on the $(6N-6)$-dimensional phase space) can be introduced for every N with well defined non-point canonical transformations (when the total angular momentum does not vanish). At the non-relativistic level, our treatment generalizes the orientation-shape bundle approach of Ref.[1].

The main results obtained can be summarized as follows: Our procedure leads to: i) The relativistic separation of the center of mass of the isolated system and the characterization of all the relevant notions of external and internal 4- and 3- centers of mass. ii) The construction of $6N-6$ canonical relative variables with respect to the internal canonical 3-center of mass for the N-body systems (unlike the non-relativistic case, they are defined by a canonical transformation which is *point* in the momenta in absence of interactions and becomes interaction-dependent when interactions are turned on). iii) The simplest construction of *Dixon multi-poles* of an isolated system with respect to a center of motion, naturally identified with the internal canonical 3-center of mass. This provides, in particular, the only way for introducing relativistic tensors of inertia.

Such results are intermediate steps in view of future developments. Actually, they furnish: iv) The natural theoretical background for a future relativistic *theory of orbits* for a N-body system, by taking into account the fact that in every instant form of relativistic dynamics, action-at-a-distance potentials appear both in the Hamiltonian and in the Lorentz boosts. v) A new definition of canonical relative variables with respect to the internal canonical 3-center of mass for every *field configuration* admitting a collective 4-vector conjugate to the conserved field 4-momentum. vi) The analysis of the relevant notions of centers of motion and associated Dixon multi-poles for an *open sub-system* of an isolated system, in a way which can be easily extended to general relativity. vii) The theoretical background for a post-Minkowskian approximation of binary systems in general relativity. The relativistic theory of orbits should provide the relativistic counterparts of the post-Keplerian parameters used in the post-Newtonian approximation [56]. viii) The theoretical background for the characterization of a *relativistic rest-frame micro-canonical mean-field thermodynamics* of N-body systems with long range interactions (Coulomb or Darwin potentials [6]) as it has already been done in Ref.[57] for non-relativistic self-gravitating and rotating systems. ix) A theoretical background that could be extended to the weak-field general relativistic N-body problem after a suitable regularization of self-energies. Actually, this framework has already been extended to charged Klein-Gordon fields interacting with the electro-magnetic field [26], Dirac fields [58] and relativistic perfect fluids [27]. Finally: x) Parametrized Minkowski theories in *non-inertial frames* are prepared for a systematic study [9] of the allowed conventions for clock synchronizations, the influence of relativistic inertial forces and the time-delays (to order $1/c^3$) for the one-way propagation of light rays (for instance between an Earth station and a satellite).

The extension of our new rotational kinematics to continuous systems like the Klein-Gordon field, an extension which should be instrumental to atomic and molecular physics at least at the classical level, is under investigation [52]. When applied to the electro-magnetic field, these methods could lead to interesting results for the problem of *phases* [53] in optics and laser physics. Also, relativistic perfect fluids have been studied in the rest-frame instant form [50] and a future application of the new rotational kinematics might give new insights for their description.

Let us conclude by noting the the non-point nature of the canonical transformations will make the quantization more difficult than in the *orientation-shape bundle* approach, where a separation of rotations from vibrations in the Schrödinger equation is reviewed in Ref.[1]. The quantizations of the original canonical relative variables and of the canonical spin bases will give equivalent quantum theories only if the non-point canonical transformations could be unitarily implementable. Up to now, these problems are completely unexplored.

[1] R.G.Littlejohn and M.Reinsch, *Gauge Fields in the Separation of Rotations and Internal Motions in the n-Body Problem*, Rev.Mod.Phys. **69**, 213 (1997).

[2] See Refs.[1, 3] for a discussion of the singular (collinear and N-body collision) configurations. Applying the SO(3) operations to any given configuration of the 3N-3 relative variables (a point in the relative configuration space) gives rise to 3 possibilities only: i) for *generic configurations* the orbit containing all the rotated copies of the configuration is a 3-dimensional manifold [diffeomorphic to the group manifold of SO(3)]; ii) for *collinear configurations* the orbit is diffeomorphic to the 2-sphere S^2; iii) for the *N-body collision configuration* (in which all the particles coincide at a single point in space) the orbit is a point.

[3] R.G.Littlejohn and M.Reinsch, *Internal or Shape Coordinates in the n-Body Problem*, Phys.Rev. **A52**, 2035 (1995).

[4] This is due to the topological complexity of the shape space generated by the singular configurations [3], which are dispersed among the generic configurations for $N \geq 4$.

[5] We shall put $c = 1$ everywhere and use the convention $\eta^{\mu\nu} = \epsilon(+---)$ for the Minkowski metric (with $\epsilon = \pm 1$ according to the particle physics or general relativity convention).

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